Artin Formalism and BSD

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Aug 2, 2024

Rank congruences

Let E be an elliptic curve defined over a number field K. Consider F/K a finite Galois extension with $G = \mathrm{Gal}(F/K)$. G acts on the points of $E(F)$ by

$$(x, y) \mapsto (\sigma(x), \sigma(y)), \quad \sigma \in G.$$

Thus $\rho = E(F) \otimes_{\mathbb{Z}} \mathbb{C}$ is a representation of G, with $\dim \rho = \mathrm{rk}\, E/F$.

For $H \leq G$, $\mathrm{rk}\, E/F^H = \dim \rho^H$.

$$\dim \rho^H = \langle \mathbb{1}, \mathrm{Res}_H^G \rho \rangle_H = \langle \mathrm{Ind}_H^G \mathbb{1}, \rho \rangle_G.$$

Thus $\mathrm{rk}\, E/F^H = \langle \mathbb{C}[G/H], \rho \rangle$.

Rank congruences

For example, let $C_p = \mathrm{Gal}(F/K)$ be an extension of number fields, E/K an elliptic curve and $\rho = E(F) \otimes_{\mathbb{Z}} \mathbb{C}$.
Let $\{\mathbb{1}, \chi_1, \dots \chi_{p-1}\}$ be the distinct characters of G, where χ_i for $i = 1, \dots p-1$ are of order p. Then

$$\begin{array}{lclcl} \mathrm{rk}\, E/K & = & \langle \mathbb{C}[G/G], \rho \rangle & = & \langle \mathbb{1}, \rho \rangle, \\ \mathrm{rk}\, E/F & = & \langle \mathbb{C}[G/C_1], \rho \rangle & = & \langle \mathbb{1} \oplus \chi_1 \oplus \cdots \oplus \chi_{p-1}, \rho \rangle. \end{array}$$

Therefore

$$\mathrm{rk}\, E/F - \mathrm{rk}\, E/K = \langle \chi_1 \oplus \cdots \chi_{p-1}, \rho \rangle$$

$$\implies \mathrm{rk}\, E/F \equiv \mathrm{rk}\, E/K \pmod{p-1}$$

L-functions

$$\text{Elliptic curve } E/K \rightsquigarrow L\text{-function } L(E/K, s)$$

with Euler product

$$L(E/K, s) = \prod_{\mathfrak{p} \subset \mathcal{O}_K} \det(1 - N(\mathfrak{p})^{-s} \operatorname{Frob}_{\mathfrak{p}}^{-1} | (V_l E^*)^{I_{\mathfrak{p}}})^{-1}.$$

Let $G = \operatorname{Gal}(F/K)$ and τ a representation of G.

$$E/K \text{ and } \tau \rightsquigarrow \text{twisted } L\text{-function } L(E/K, \tau, s)$$

with Euler product

$$L(E/K, \tau, s) = \prod_{\mathfrak{p} \subset \mathcal{O}_K} \det(1 - N(\mathfrak{p})^{-s} \operatorname{Frob}_{\mathfrak{p}}^{-1} | (\tau \otimes V_l E^*)^{I_{\mathfrak{p}}})^{-1}.$$

These L-functions satisfy the following properties, known as *Artin formalism*

- $L(E/K, \tau \oplus \tau', s) = L(E/K, \tau, s) L(E/K, \tau', s)$,
- $L(E/K, \mathbb{C}[G/H], s) = L(E/F^H, s)$ for $H \leq G$.

Th

BSD 1

Conjecture (BSD 1)

If E is an elliptic curve over a number field K, then

$$\operatorname{rk} E/K = \operatorname{ord}_{s=1} L(E/K, s).$$

Conjecture (BSD Analogue for twists, [Deligne-Gross])

Let E be an elliptic curve over a number field K. If F/K is a finite Galois extension of number fields with $G = \operatorname{Gal}(F/K)$, then

$$\langle \tau, E(F) \otimes_{\mathbb{Z}} \mathbb{C} \rangle = \operatorname{ord}_{s=1} L(E/K, \tau, s)$$

for τ a representation of G.

BSD 2

Conjecture (BSD 2)

Let E be an elliptic curve over a number field K. The group $\text{Ш}_{E/K}$ has finite order and the leading term of the Taylor series at $s = 1$ of the L-function is

$$\lim_{s \to 1} \frac{L(E/K, s)}{(s-1)^r} \cdot \frac{\sqrt{|\Delta_K|}}{\Omega_+(E)^{r_1+r_2}|\Omega_-(E)|^{r_2}} = \underbrace{\frac{\mathrm{Reg}_{E/K}\,|\text{Ш}_{E/K}|\,C_{E/K}}{|E(K)_{\mathrm{tors}}|^2}}_{:=\mathrm{BSD}(E/K)}$$

If E/K is semistable

$$C_{E/K} = \prod_{\mathfrak{p} \subset \mathcal{O}_K} c_{\mathfrak{p}}(E/K),$$

where $c_{\mathfrak{p}}(E/K) = [E(K_{\mathfrak{p}})\colon E_0(K_{\mathfrak{p}})]$ is the local Tamagawa number. Else, there are some extra factors at the primes of additive reduction.

Question: Can we factor $\mathrm{BSD}(E/K)$ according to Artin representations?

BSD analogue for twists

Conjecture (BSD-term conjecture, [Dokchitser-Evans-Wiersema 21])

Let $E/\mathbb{Q}$ be an elliptic curve **over the rationals**, and $G = \mathrm{Gal}(F/\mathbb{Q})$. For each representation τ of G there exists an invariant $\mathrm{BSD}(E,\tau) \in \mathbb{C}^\times$ such that:

1. $\mathrm{BSD}(E, \tau \oplus \tau') = \mathrm{BSD}(E,\tau)\,\mathrm{BSD}(E,\tau')$, where τ' is a rep. of G,
2. $\mathrm{BSD}(E, \mathbb{C}[G/H]) = \mathrm{BSD}(E/F^H)$ for $H \leq G$.

If in addition $\langle \tau, E(F) \otimes_{\mathbb{Z}} \mathbb{C} \rangle = 0$, then

3. Galois equivariance: $\mathrm{BSD}(E,\tau) \in \mathbb{Q}(\tau)^\times$ and

$$\mathrm{BSD}(E, \tau^{\mathfrak{g}}) = \mathrm{BSD}(E,\tau)^{\mathfrak{g}} \text{ for } \mathfrak{g} \in \mathrm{Gal}(\mathbb{Q}(\tau)/\mathbb{Q}).$$

D_{10} example

Consider $G = \mathrm{Gal}(F/\mathbb{Q}) = D_{10} = \langle x, y \mid y^2 = x^5 = e, yxy = x^{-1} \rangle$.

	e	x	yx	yx^2
$\mathbb{1}$	1	1	1	1
ε	1	-1	1	1
τ	2	0	$\frac{-1-\sqrt{5}}{2}$	$\frac{-1+\sqrt{5}}{2}$
τ^σ	2	0	$\frac{-1+\sqrt{5}}{2}$	$\frac{-1-\sqrt{5}}{2}$

Table: Character table of G with $\mathrm{Gal}(\mathbb{Q}(\sqrt{5})/\mathbb{Q}) = \langle \sigma \rangle$.

One has $\mathbb{C}[G/C_2] \simeq \mathbb{C}[G/G] \oplus \tau \oplus \tau^\sigma$. Let $L = F^{C_2}$.

Let E be an elliptic curve over $\mathbb{Q}$ such that $\langle E(F) \otimes_{\mathbb{Z}} \mathbb{C}, \tau \rangle = 0$. Assuming the BSD-term conjecture,

$$\frac{\mathrm{BSD}(E/L)}{\mathrm{BSD}(E/\mathbb{Q})} = \mathrm{BSD}(E,\tau) BSD(E,\tau^\sigma) = N(\mathrm{BSD}(E,\tau)),$$

hence is of the form $x^2 - 5y^2$ for x, $y \in \mathbb{Q}$.

D_{10} example

Suppose $\mathrm{rk}\, E/F = 0$. Then

$$\frac{\mathrm{BSD}(E/L)}{\mathrm{BSD}(E/\mathbb{Q})} = \frac{|E(\mathbb{Q})_{\mathrm{tors}}|^2 |\text{Ш}_{E/L}| C_{E/L}}{|E(L)_{\mathrm{tors}}|^2 |\text{Ш}_{E/\mathbb{Q}}| C_{E/\mathbb{Q}}} \equiv \frac{C_{E/L}}{C_{E/\mathbb{Q}}} \pmod{\mathbb{Q}^{\times 2}}$$

Suppose E has semistable reduction over $\mathbb{Q}$. Then $\frac{C_{E/L}}{C_{E/\mathbb{Q}}} = \prod_p \frac{\prod_{w|p} c_w(E/L)}{c_p(E/\mathbb{Q})}$.

- If $E/\mathbb{Q}_p$ has good reduction, $\frac{\prod_{w|p} c_w(E/L)}{c_p(E/\mathbb{Q})} = 1$.
- If $E/\mathbb{Q}_p$ has split multiplicative reduction with $c_p(E/\mathbb{Q}) = n$,

$$\frac{\prod_{w|p} c_w(E/L)}{c_p(E/\mathbb{Q})} = \frac{\prod_{w|p} e_{w/p} \cdot n}{n}.$$

  - If p is totally ramified, this is $5 = (2+\sqrt{5})(2-\sqrt{5})(-5)$,
  - If p has decomp. group C_5 and is unramified, this is 1,
  - If p has decomp. group = inertia group = C_2, this is $(2n)^2$.

D_{24} example

Consider $G = \mathrm{Gal}(F/\mathbb{Q}) = D_{24}$. This has two Galois conjugate 2-dim. irreps. τ, τ^σ with $\mathbb{Q}(\tau) = \mathbb{Q}(\sqrt{3})$. We have

$$\mathbb{C}[G/C_2] \oplus \mathbb{C}[G/D_6] \simeq \mathbb{C}[G/C_2 \times C_2] \oplus \mathbb{C}[G/S_3] \oplus \tau \oplus \tau^\sigma.$$

If $E/\mathbb{Q}$ satisfies $\langle E(F) \otimes_{\mathbb{Z}} \mathbb{C}, \tau \rangle = 0$ and we assume that $\mathrm{rk}\, E/F = 0$, the conjecture implies that

$$\frac{\mathrm{BSD}(E/F^{C_2})\,\mathrm{BSD}(E/F^{D_6})}{\mathrm{BSD}(E/F^{C_2 \times C_2})\,\mathrm{BSD}(E/F^{S_3})} = x^2 - 3y^2, \quad x, y \in \mathbb{Q}.$$

But if $E/\mathbb{Q}$ has additive reduction of Type II at 11 and good reduction elsewhere, this product is $11 \cdot \square$, which is **not** a norm from $\mathbb{Q}(\sqrt{3})$.

Thus our assumption that $\mathrm{rk}\, E/F = 0$ is false and so $\mathrm{rk}\, E/F > 0$.
(In this case one can use root numbers to show that the rank must grow. Assuming $F/\mathbb{Q}$ is totally real and adding the product of regulators, one gets $22 \cdot \square = (5 + \sqrt{3})(5 - \sqrt{3}) \cdot \square$.)

Norm Relations Test

Theorem (Norm Relations Test, [Dokchitser-Evans-Wiersema 21])

Suppose the BSD-term conjecture holds. Consider $E/\mathbb{Q}$, and $F/\mathbb{Q}$ with $G = \mathrm{Gal}(F/\mathbb{Q})$. Let ρ be a rep. of G with $\mathbb{Q}(\rho) = \mathbb{Q}(\sqrt{D})$ and $\mathrm{Gal}(\mathbb{Q}(\sqrt{D})/\mathbb{Q}) = \langle\sigma\rangle$. Consider

$$(\rho \oplus \rho^{\sigma})^{\oplus m} \oplus \bigoplus_j \mathbb{C}[G/H_j'] \simeq \bigoplus_i \mathbb{C}[G/H_i]$$

for some $m \geq 1$ and $H_i, H_j' \leq G$. If

$$\frac{\prod_i C_{E/F^{H_i}}}{\prod_j C_{E/F^{H_j'}}} \notin \begin{cases} (\mathbb{Q})^{\times 2} & m \text{ even}, \\ N_{\mathbb{Q}(\sqrt{D})/\mathbb{Q}}(\mathbb{Q}(\sqrt{D})^{\times}) & m \text{ odd}, \end{cases}$$

then $\mathrm{rk}\, E/F > 0$.

Our work

- With Albert Lopez Bruch, we've shown cases where this test cannot be used to force positive rank. This is the case when G is cyclic or of odd order.
- We're currently working on proving that once one adds in the appropriate product of regulators, one does get a norm from the relevant quadratic field.
- We are trying to prove this algebraically, assuming BSD and the parity conjecture for twists. This contrasts the work of Dokchitser-Evans-Wiersema where this is a consequence of their BSD-term conjecture.

Thank you!