# Reduction types of genus 2 curves

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# Elliptic curves

Let $E$ be an elliptic curve over a number field $K$. At a prime $\mathfrak{p}$, $E$ has either

- Semistable reduction: good or multiplicative,
- Non-semistable reduction: additive.

**Deligne-Mumford theorem:** Non-semistable reduction becomes semistable after a finite ramified extension.

## Example

Let $E\colon y^2 = x^3 - p^2$. This has additive reduction at $p$. Attains good reduction at the prime above $p$ in $\mathbb{Q}(\sqrt[3]{p})$.

Additive reduction can be split into two types: potentially good reduction and potentially multiplicative reduction.

## Definition (Reduction type)

Let $C/K$ be a (nice) curve over a number field $K$. The **reduction type** of $C$ at a prime $\mathfrak{p}$ of $K$ is the structure of the special fibre of the minimal regular model of $C$ over $K_{\mathfrak{p}}^{\mathrm{nr}}$.

E.g. for elliptic curves these are known as Kodaira types, and they come in ten families.

# Art in mathematics

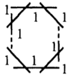
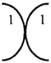
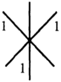
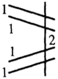
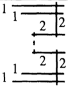
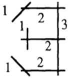
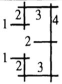
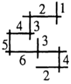
| Kodaira symbol | $\mathrm{I}_0$ | $\mathrm{I}_n$ $(n \geq 1)$ | II | III | IV | $\mathrm{I}_0^*$ | $\mathrm{I}_n^*$ $(n \geq 1)$ | $\mathrm{IV}^*$ | $\mathrm{III}^*$ | $\mathrm{II}^*$ |
|---|---|---|---|---|---|---|---|---|---|---|
| Special fiber $\tilde{\mathcal{C}}$ (The numbers indicate multiplicities) |  |  |  |  |  |  |  |  |  |  |
| $m =$ number of irred. components | 1 | $n$ | 1 | 2 | 3 | 5 | $5+n$ | 7 | 8 | 9 |
| $E(K)/E_0(K) \cong \tilde{\mathcal{E}}(k)/\tilde{\mathcal{E}}^0(k)$ | (0) | $\frac{\mathbb{Z}}{n\mathbb{Z}}$ | (0) | $\frac{\mathbb{Z}}{2\mathbb{Z}}$ | $\frac{\mathbb{Z}}{3\mathbb{Z}}$ | $\frac{\mathbb{Z}}{2\mathbb{Z}} \times \frac{\mathbb{Z}}{2\mathbb{Z}}$ | $\frac{\mathbb{Z}}{2\mathbb{Z}} \times \frac{\mathbb{Z}}{2\mathbb{Z}}$ $n$ even; $\frac{\mathbb{Z}}{4\mathbb{Z}}$ $n$ odd | $\frac{\mathbb{Z}}{3\mathbb{Z}}$ | $\frac{\mathbb{Z}}{2\mathbb{Z}}$ | (0) |
| $\tilde{\mathcal{E}}^0(k)$ | $\tilde{E}(k)$ | $k^*$ | $k^+$ | $k^+$ | $k^+$ | $k^+$ | $k^+$ | $k^+$ | $k^+$ | $k^+$ |
| **Entries below this line only valid for** $\mathrm{char}(k) = p$ **as indicated** | | | | | | | | | | |
| $\mathrm{char}(k) = p$ | | | $p \neq 2,3$ | $p \neq 2$ | $p \neq 3$ | $p \neq 2$ | $p \neq 2$ | $p \neq 3$ | $p \neq 2$ | $p \neq 2,3$ |
| $v(\mathcal{D}_{E/K})$ (discriminant) | 0 | $n$ | 2 | 3 | 4 | 6 | $6+n$ | 8 | 9 | 10 |
| $f(E/K)$ (conductor) | 0 | 1 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| behavior of $j$ | $v(j) \geq 0$ | $v(j) = -n$ | $\tilde{j} = 0$ | $\tilde{j} = 1728$ | $\tilde{j} = 0$ | $v(j) \geq 0$ | $v(j) = -n$ | $\tilde{j} = 0$ | $\tilde{j} = 1728$ | $\tilde{j} = 0$ |

Table 4.1: A Table of Reduction Types

# Tate's algorithm

For elliptic curves, one can determine the reduction type via **Tate's algorithm**.

## Corollary (of Tate's algorithm)

*Let $E/K$ be an elliptic curve, $\mathfrak{p}$ a prime of $K$ such that $K_\mathfrak{p}$ has residue characteristic $\geq 5$. Then the reduction type of $E$ at $\mathfrak{p}$ is determined by $v_\mathfrak{p}(\Delta_E)$ and $v_\mathfrak{p}(j_E)$.*

Easy to see how the reduction type changes in ramified extensions: look at $v_\mathfrak{p}(\Delta_E)$ mod 12 and $v_\mathfrak{p}(j_E)$.

## Example

$E\colon y^2 = x^3 - p^2$ has $v_p(\Delta_E) = 4 \implies$ type $\mathrm{IV}$ reduction at $p$.
Consider an extension $K/\mathbb{Q}$, $\mathfrak{p}$ a prime above $p$ with ramification degree $e$.

$$\text{Reduction type of } E \text{ at } \mathfrak{p} = \begin{cases} \text{Type } \mathrm{I}_0 & e \equiv 0 \mod 3, \\ \text{Type IV} & e \equiv 1 \mod 3, \\ \text{Type IV}^* & e \equiv 2 \mod 3. \end{cases}$$

# Higher genus curves

- As the genus grows, so does the number of reduction types.
- For genus 2 curves, reduction types occur in 104 families. For genus 3 curves, there's 1901 families!
- Reduction types of genus 2 curves have been classified by Namikawa and Ueno.
- Liu and Mestre: determine the reduction type of a genus 2 curve from the Igusa invariants (+ some other invariants). But this doesn't illuminate how the reduction type changes in ramified extensions.
- All genus 2 curves are hyperelliptic, i.e. can be expressed as the smooth projective curve associated to the affine equation

  $$y^2 = f(x)$$

  where $f(x)$ has degree $5/6$.
- **Our aim:** Use cluster pictures to understand reduction type.

# Cluster pictures

Let $y^2 = f(x)$ be a hyperelliptic curve. Its associated *cluster picture* displays the $p$-adic distances between the roots of $f(x)$.

## Example

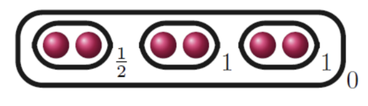
Let $C\colon y^2 = (x^2 - p)((x-1)^2 + p^2)((x+1)^2 - p^2)$ over $\mathbb{Q}_p$. The roots of $f$ are

$$\mathcal{R} = \{\pm\sqrt{p}, 1 \pm ip, -1 \pm p\}.$$

One has

$$\begin{array}{lll} v(\sqrt{p} - (-\sqrt{p})) = \frac{1}{2}, & v(\pm\sqrt{p} - (1 \pm ip)) = 0, & v(\pm\sqrt{p} - (-1 \pm p)) = 0, \\ v(1 + ip - (1 - ip)) = 1, & v(1 \pm ip - (-1 \pm p)) = 0, & v(-1 + p - (-1 - p)) = 1. \end{array}$$

The cluster picture is



# Cluster pictures

## Definition

Let $p$ be an odd prime, $K$ a $p$-adic field and $C/K$ a hyperelliptic curve given by

$$y^2 = f(x) = c(x - a_1)(x - a_2)\cdots(x - a_n)$$

Let $\mathcal{R} = \{a_1, \ldots a_n\}$ denote the set of roots of $f$. Let $\mathfrak{p}$ be the unique prime of $\mathcal{O}_K$. We define the cluster picture $\Sigma_{\mathfrak{p}}$ associated to $C$ with respect to $\mathfrak{p}$ as

$$\Sigma_{\mathfrak{p}} := \{\mathfrak{s} \in \mathcal{P}(\mathcal{R}) |\ \mathfrak{s} = D_{z,d} \cap \mathcal{R} \text{ for some } z \in \overline{\mathrm{K}},\ d \in \mathbb{Q}\},$$

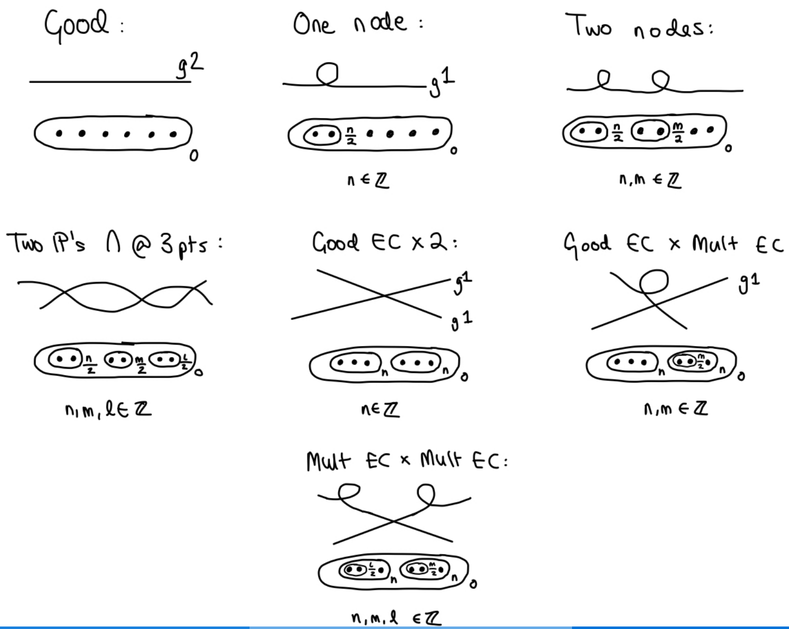
where $D_{z,d} := \{x \in \overline{\mathrm{K}} \mid v_{\mathfrak{p}}(x - z) \geq d\}$.

i.e. $\Sigma_{\mathfrak{p}}$ are the subsets of $\mathcal{R}$ which are cut out by bounded $p$-adic discs.

## Theorem (Farragi-Nowell)

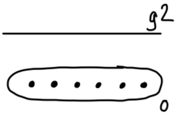
*Let $K$ be a $p$-adic field with $p$ odd. Let $C\colon y^2 = f(x)$ be a hyperelliptic curve over $K$ with tame potentially semistable reduction. Then the reduction type is determined by the cluster picture of $C$ and the valuation of the leading coefficient of $f$.*

# Genus 2 semistable reduction types

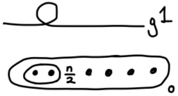


## Genus 2 semistable reduction types

Good: 18 types

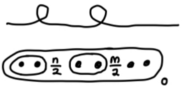



One node: 12 types



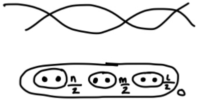

$n \in \mathbb{Z}$

Two nodes: 5 types



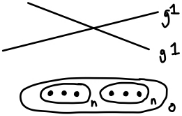

$n, m \in \mathbb{Z}$

Two $\mathbb{P}$'s ∩ @ 3pts: 6 types



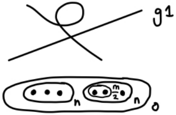

$n, m, \ell \in \mathbb{Z}$

Good EC x 2: 42 types



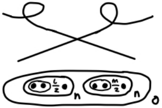

$n \in \mathbb{Z}$

Good EC x Mult EC: 16 types



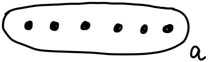

$n, m \in \mathbb{Z}$

Mult EC x Mult EC: 5 types

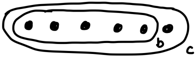


# Potentially good reduction

For example, if $C : y^2 = f(x)$ has $\deg f = 6$ and potentially good reduction, then its cluster picture must be of the form




$a \in \{0, 1/6, 1/3, 1/2, 2/3, 5/6, 1/5, 2/5, 3/5, 4/5,\} \bmod 1$




$b \in \{0, 1/4, 1/2, 3/4, 1/5, 2/5, 3/5, 4/5\} \bmod 1, \quad c \in \mathbb{Z}$

One determines the reduction type in the first case by ($a \mod 1$, $v(c_f) \mod 2$), and in the second case by $b \mod 2$.

# Strategy

- Group reduction types into those with same potential stable model.
- For each grouping, describe the possible cluster pictures that can occur.
- Choose a canonical cluster picture for each reduction type, in a way that behaves nicely with taking ramified extensions.
- Tabulate it!

# The End

**Thank you!**