Introduction to étale cohomology

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In this note, we try to motivate étale cohomology, and introduce étale morphisms, which will be needed to define étale cohomology in the next talk. At the end there is some discussion on the similarities of étale cohomology with Galois cohomology. Throughout we will recall some necessary algebraic geometry concepts.

For sources, I consulted Chapters 3,6 of [Poo17], [Con16], the stacks project [Aut] for random definitions, Milne's notes on étale cohomology [Mil13], and lastly math overflow.

1 Why study étale cohomology?

1.1 Slandering the Zariski topology

A main motivation for introducing étale cohomology was to construct a *Weil cohomology theory* in order to prove the Weil conjectures, though I won't say anything on this.

Another means of motivating étale cohomology is by explaining the shortcomings of usual sheaf cohomology for schemes. Let X be a scheme, and $\mathcal{C}$ the category with objects the Zariski opens of X and morphisms given by inclusions of open sets. Recall that a sheaf is a functor $\mathcal{F}\colon \mathcal{C}^{\mathsf{op}} \to \mathbf{Set}$ (or $\mathbf{Grp}$ etc) satisfying some additional properties. One obtains sheaf cohomology by resolving the global sections functor

$$\mathcal{F} \mapsto \mathcal{F}(X)$$

which is left exact, but not right exact.

The problem with sheaf cohomology is that it too often gives us uninteresting cohomology. Consider the following example:

Example 1.1 (Flasque sheaves)**.** Let X be an irreducible variety over $\mathbb{C}$. Consider the constant sheaf $\underline{A}$ for some abelian group A. Now irreducibility implies that for $U \subset X$ open and non-empty, $\underline{A}(U) = A$ (since U is connected). Then $\underline{A}(\emptyset) = 0$ and so restriction maps are always surjective. This means that $\underline{A}$ is flasque, and general lore tells us that $H^i(X, \underline{A}) = 0$ for $i > 0$.

On the other hand, if X is a smooth projective complex variety, then $X(\mathbb{C})$ is a complex manifold, and one has that $H^{2\dim X}_{\text{sing}}(X(\mathbb{C}), \mathbb{Z}) = \mathbb{Z}$. ♢

The thing that one is meant to say went wrong in this example is that the Zariski topology has too few open sets (and they're too big). The insight of Grothendieck was that by redefining our idea of open sets, one could obtain a better cohomology.

1.2 Motivation via homological algebra

Let G be a finite group, M a G-module. Recall our recipe for computing $H^i(G,M)$:

1. Compute a projective resolution of $\mathbb{Z}$,
2. apply $\mathrm{Hom}_{\mathbb{Z}[G]}(-,M)$,
3. take homology of the chain.

This is the recipe for computing Ext groups, which are "the derived functors of Hom". One has $H^*(G,M) = \mathrm{Ext}^*_{\mathbb{Z}[G]}(\mathbb{Z},M)$ where $\mathbb{Z}$ is the trivial $\mathbb{Z}[G]$-module. The point is that group cohomology emerges by rectifying the failure of right exactness for the functor $B \mapsto B^G = \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z},B)$, where B is a $\mathbb{Z}[G]$-module.

Other cohomology theories can be described as Ext groups also. For example, if X is a topological space, then the sheaf cohomology of a sheaf A of abelian groups on X can be expressed as $H^*(X,A) = \mathrm{Ext}^*(\mathbb{Z}_X, A)$. Here we are working in the abelian category of sheaves of abelian groups on X, and $\mathbb{Z}_X$ is the sheaf of locally constant $\mathbb{Z}$-valued functions. This time you took an injective resolution of your sheaf to compute sheaf cohomology, but this is just another way to compute Ext groups (the point being your category may not have enough projectives).

Ultimately, we will be able to write étale cohomology as Ext groups: $H^*(X_{\text{ét}},-) = \mathrm{Ext}_X(\mathbb{Z},-)$ (we take derived functors of the global sections functor, as in sheaf cohomology). Perhaps this gives some reasoning for why when $X = \operatorname{Spec} k$ for k a field, the étale cohomology groups are isomorphic to $H^*(G_k, F)$ (for a certain G_k module F). We will see this is the case because we are resolving essentially the same functor.

2 Étale morphisms

The first step in defining étale cohomology is by replacing the category of open sets of a scheme X with the category $\text{Ét}(X)$.

Definition 2.1. For a scheme X, $\text{Ét}(X)$ is the category whose objects are étale morphisms $U \to X$ where U is a scheme, and whose morphisms are X-morphisms, i.e. such that the following commutes

$$\begin{tikzcd} U \arrow[rr] \arrow[dr] & & V \arrow[dl] \\ & X & \end{tikzcd}$$

To understand that, we first need to learn what étale morphisms are. One can view étale morphisms for schemes to as the analogue of local homeomorphisms for complex manifolds. The usual topology for schemes is the Zariski topology, and so one may guess that the obvious analogue would be:

Non-analogue: Let X, Y be schemes, $f\colon X \to Y$ a morphism of schemes. Then f is a local isomorphism if any $x \in X$ has an open (with respect to the Zariski topology) neighborhood U such that f is an isomorphism of schemes onto its image.

What's the problem with this? The problem is that the open sets in the Zariski topology are too big.

Example 2.2. Let k be an algebraically-closed field, and $k^\times = \operatorname{Spec} k[t, t^{-1}]$. Then the map $f\colon k^\times \to k^\times$ given by $z \mapsto z^2$ is not a local isomorphism in the Zariski sense. Indeed open subsets of $k^\times$ are $k^\times \setminus \{\text{finitely many points}\}$, and these do not map isomorphically to their image. $\diamondsuit$

Étale morphisms are a good analogue because the following holds:

Lemma 2.3 ([Con16, Lemma 1.1.1.6])**.** *A map $f\colon X \to S$ of $\mathbb{C}$-schemes is an étale morphism if and only if f^{an} is a local isomorphism of analytic spaces.*

Now let's get into the definitions. The following condition keeps things non-pathological:

Definition 2.4 ([Con16, Definition 1.1.1.1])**.** A morphism $f\colon X \to Y$ of schemes is locally of finite presentation if there is an affine open covering $\{\operatorname{Spec} B_i\}$ of Y and an affine open covering $\{A_{ij}\}$ of each $f^{-1}(\operatorname{Spec} B_i)$ such that each A_{ij} is of the form $A_{ij} = B_i[T_1, \dots T_n]/I$ for I a finitely generated ideal.

Remark 2.5. If f is locally of finite presentation then it is locally of finite type. If Y is a locally Noetherian schemes then these two notions coincide. $\diamondsuit$

Example 2.6. The morphism $f\colon \operatorname{Spec} \mathbb{Q} \to \operatorname{Spec} \mathbb{Z}$ is not locally of finite presentation. $\diamondsuit$

To define an étale morphism one needs to define a flat and unramified morphism. We deal with flatness first. Recall that if $f\colon X \to Y$ is a map of schemes and $x \in X$, then $\mathcal{O}_{X,x}$ has the structure of a $\mathcal{O}_{Y,f(x)}$-module. Indeed, the map $f^\#\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$ induces

$$f^\#_x\colon \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}, \quad (U, s) \mapsto \left(f^{-1}(U), f^\#_U(s)\right).$$

Therefore one can consider $\mathcal{O}_{X,x}$ as a $\mathcal{O}_{Y,f(x)}$-module with multiplication through $f^\#_x$.

Definition 2.7 (Flat morphism)**.** A map $f\colon X \to Y$ of schemes is flat at a point $x \in X$ if $\mathcal{O}_{X,x}$ is flat as an $\mathcal{O}_{Y,f(x)}$-module (i.e. tensoring by $\mathcal{O}_{X,x}$ preserves exact sequences of $\mathcal{O}_{Y,f(x)}$-modules). The map f is flat if it is flat at every $x \in X$.

Remark 2.8. Let $A \to B$ be a homomorphism of commutative rings. Then $\operatorname{Spec} B \to \operatorname{Spec} A$ is flat if and only if B is flat over A. $\diamondsuit$

Example 2.9. Consider the map $k[t]/(t^2) \to k$ given by $a + bt \mapsto t$. Then the corresponding map $\operatorname{Spec} k \to \operatorname{Spec} k[t]/(t^2)$ is not flat. Indeed k is not flat as a $k[t]/(t^2)$ module, which one can see by tensoring the exact sequence $0 \to (t) \to k[t]/(t^2) \to k \to 0$ by $k = (k[t]/(t^2))/(t)$. $\diamondsuit$

By taking the fibres of a flat morphism, one gets a flat family of schemes. One can determine the dimension of the fibres as follows:

Proposition 2.10 ([Har77, Ch. III, Proposition 9.5])**.** *Let $f\colon X \to Y$ be a flat morphism of schemes of finite type over a field k. For any $x \in X$, let $y = f(x)$. Then*

$$\dim_x(X_y) = \dim_x X - \dim_y Y,$$

where $\dim_x X$ is the dimension of the local ring $\mathcal{O}_{X,x}$ and X_y is the (scheme-theoretic) fibre over y.

In nicer situations, this tells us that the dimension of the fibres should remain constant:

Corollary 2.11 ([Har77, Ch. III, Corollary 9.6]). *Let $f\colon X \to Y$ be a flat morphism of schemes of finite type over a field k, and assume that Y is irreducible. Assume that the dimension of the irreducible components of X are equal. Then*

$$\dim X_y = \dim X - \dim Y$$

for all $y \in Y$ with $X_y \neq \emptyset$. In particular $\dim X_y$ is independent of y.

Example 2.12.

1. Let $X = \operatorname{Spec}(k[x,y,t]/(xy-t))$, $Y = \operatorname{Spec}(k[t])$ and consider the map $X \to Y$ induced by the natural map $k[t] \to k[x,y,t]/(xy-t)$. Note that X and Y are regular over k. This is flat. One can see that the fibres are irreducible hyperbolas when $t \neq 0$ and becomes the reducible scheme consisting of two lines at $t = 0$.

2. The family of curves $y^2 = x^3+x^2+tx$ is a flat family over $\operatorname{Spec} k[t]$. At $t = 0$ it degenerates to the nodal cubic.

$\diamondsuit$

Example 2.13. On the other hand, consider $X = \operatorname{Spec}(k[x,y,t]/(txy-t))$ and $Y = \operatorname{Spec}(k[t])$ and the obvious map $X \to Y$. Then for $y \neq 0$ one has that X_y is the hyperbola $xy = 1$. But at $t = 0$ we get the affine plane, hence this map is not flat. $\diamondsuit$

Now we turn to unramified morphisms. To define these we first need to define unramified on the level of rings:

Definition 2.14 ([Poo17, Definition 3.5.28]). A local homomorphism $g\colon A \to B$ of local rings (i.e. $g^{-1}(\mathfrak{m}_B) = \mathfrak{m}_A$) is unramified if

1. $g(\mathfrak{m}_A)B = \mathfrak{m}_B$,
2. $B/\mathfrak{m}_B$ is finite and separable over $A/\mathfrak{m}_A$.

Definition 2.15 (Unramified morphism). A morphism $f\colon X \to Y$ of schemes is unramified if it is locally of finite presentation and if the maps $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ are unramified for all $x \in X$.

Remark 2.16. A morphism of schemes $f\colon X \to Y$ is unramified if and only if it is locally of finite presentation and $\Omega_{X/Y} = 0$ (cotangent sheaf). $\diamondsuit$

Example 2.17. Let L/K be a finite extension of number fields and consider the map $f\colon \operatorname{Spec}\mathcal{O}_L \to \operatorname{Spec}\mathcal{O}_K$. This sends a prime ideal $\mathfrak{q}$ in $\mathcal{O}_L$ to the prime ideal $\mathfrak{p}$ lying below it. Then f is unramified at $\mathfrak{q}$ if and only if $\mathfrak{q}$ is unramified in L/K in the usual sense. Indeed $\mathfrak{p}\mathcal{O}_{L,\mathfrak{q}} = (\mathfrak{p}\mathcal{O}_L)_{\mathfrak{q}} = (\mathfrak{q}\mathcal{O}_{L,\mathfrak{q}})^{e_{\mathfrak{q}/\mathfrak{p}}}$ $\diamondsuit$

Example 2.18. Consider the squaring map $\mathbb{A}^1_k \to \mathbb{A}^1_k$. Then this is unramified away from the origin, assuming 2 is invertible in k. $\diamondsuit$

Example 2.19. Consider the normalization map $\mathbb{A}^1_k \to X$ where $X = \operatorname{Spec}(k[x,y]/(y^2-x^3))$ and assume $\operatorname{char} k \neq 2$. Recall this comes from the ring map $(x,y) \mapsto (t^3,t^2)$. We can determine when $\Omega_{X/Y}$ is non-zero by computing the relative Kähler differential module. One can write $X = \operatorname{Spec} A$ for $A = k[t^2,t^3]$ and $\mathbb{A}^1_k = \operatorname{Spec} B$ for $B = A[x]/(x^2-t^2)$. Then $\Omega_{B/A}$ is generated by dx subject to the relation $0 = d(x^2-t^2) = 2xdx$. So as a B-module $\Omega_{B/A}$ is isomorphic to $Bdx/B(2xdx) = A[x]/(2x, x^2-t^2) = k[t]/(t)$. We are unramfied at $x \in X$ if $(\Omega_{B/A})_x = 0$, hence unramified away from zero (since $(k[t]/(t))_{\mathfrak{p}} = k[t]_{\mathfrak{p}} \otimes_{k[t]} k[t]/(t) = k[t]_{\mathfrak{p}}/(t)k[t]_{\mathfrak{p}} = 0$ unless $\mathfrak{p} = (t)$). This is what one would expect. $\diamondsuit$

Definition 2.20 ([Poo17, Definition 3.5.34])**.** A morphism $f \colon X \to Y$ is étale at a point $x \in X$ if it is flat at x and unramified at x. The map f is étale if it is étale at every $x \in X$.

Proposition 2.21 ([Mil13, Proposition 2.11])**.** *The following are some properties of étale morphisms:*

1. *Open immersions are étale morphisms,*
2. *Compositions of étale morphisms are étale morphisms,*
3. *Base change of an étale morphism is an étale morphism,*
4. *If $\varphi \circ \psi$ and φ are étale, then so is ψ.*

The above properties also hold if one replaces étale morphisms by flat morphisms or unramified morphisms. The following more explicit definition can be helpful for determining when a morphism is étale.

Proposition 2.22 (Standard étale morphisms, [Mil13, Ch1, P22])**.** *A morphism $f \colon X \to Y$ is étale if and only if it is locally standard étale.*

This means that, for each $x \in X$, $y = f(x)$, there exists affine opens $x \in U$, $y \in V$ such that $f(U) \subseteq V$ and we have

$$V = \operatorname{Spec} R, \quad U = \operatorname{Spec} R[x]_h/(g)$$

where $h, g \in R[x]$ are polynomials, $R[x]_h$ is the localization at h, and g is monic and the derivative g' is a unit in $R[x]_h/(g)$.

Example 2.23. $\mathbb{G}_m \to \mathbb{G}_m$ induced by $t \mapsto t^n$, where $\mathbb{G}_m = \operatorname{Spec} k[t, t^{-1}]$ is an étale morphism, if n is prime to the characteristic of k . Indeed, one can view this as induced by the map $k[t, t^{-1}] \to k[t, t^{-1}, u]/(u^n - t) \simeq k[u, u^{-1}]$ and the derivative of $u^n - t$ is nu^{n-1} which is a unit. $\diamondsuit$

Example 2.24 (Separable extensions)**.** Let $Y = \operatorname{Spec} k$ where k is a field, and let X be of finite type over k. Then flatness of the map $f \colon X \to Y$ is automatic. If $X = \operatorname{Spec} L$ where L/k is a finite separable extension of k, then f is unramified. Indeed, write $L = k[x]/(f)$. Then $\Omega_{L/k} \simeq k[t]/(f, f')dt = 0$ since f and f' are coprime.

More generally, $X \to Y$ is étale, if and only if X is a finite product of $\operatorname{Spec} L_i$ where L_i/k are finite separable extensions. $\diamondsuit$

3 Appetite whetting: étale cohomology and Galois cohomology

Now let $X = \operatorname{Spec} k$, where k is a field. Then if $U \to X$ is an étale morphism, one has that U is a disjoint union of k-affine schemes $\operatorname{Spec} L$ where L/k is a finite separable extension.

Consider a functor $\mathcal{F} \colon \text{Ét}(X)^{\mathsf{op}} \to \mathbf{Set}$ (i.e. a presheaf). For the value of $\mathcal{F}$ on the morphism $U \to X$ we write $\mathcal{F}(U)$. It is a sheaf (as will be properly defined in the next talk) if and only if the following properties are satisfied:

1. For any disjoint union $\coprod U_i$ we have

$$\mathcal{F}(\coprod U_i) = \prod \mathcal{F}(U_i).$$

2. For all finite separate extensions $k''/k'/k$ such that k''/k' is Galois we have that $\mathcal{F}(\operatorname{Spec} k') \to \mathcal{F}(\operatorname{Spec} k'')$ is injective and $\mathcal{F}(\operatorname{Spec} k') = \mathcal{F}(\operatorname{Spec} k'')^{\operatorname{Gal}(k''/k')}$.

Write $\mathcal{F}(L)$ for $\mathcal{F}(\operatorname{Spec} L)$. Fix a separable closure k_s and define $\mathcal{F}(k_s) := \varinjlim \mathcal{F}(L)$ ranging over all finite separable Galois extensions L/k contained in k_s. Then G_k acts on $\mathcal{F}(k_s)$ (this follows from functoriality of $\mathcal{F}$ and Spec, so that the action of G_k on the $\mathcal{F}(L)$ is compatible).

Theorem 3.1 ([Poo17, Theorem 6.4.6])**.** *Let k be a field and choose a separable closure k_s of k.*

1. *The functor*

$$\begin{aligned} \{\textit{sheaves of sets on } (\operatorname{Spec} k)_{\acute{e}t}\} &\to \{G_k\textit{-sets}\} \\ \mathcal{F} &\mapsto \mathcal{F}(k_s) \end{aligned}$$

is an equivalence of categories. The global section functor corresponds to the functor that takes a G_k-set M to M^{G_k}.

2. *This restricts to an equivalence*

$$\{\textit{abelian sheaves on } (\operatorname{Spec} k)_{et}\} \to \{G_k\textit{-modules}\}.$$

3. *There are natural isomorphisms*

$$H^q_{\acute{e}t}(\operatorname{Spec} k, \mathcal{F}) \simeq H^q(G_k, \mathcal{F}(k_s))$$

for all $q \in \mathbb{N}$.

Proof sketch.

1. The sheaf axioms show that there is a well-defined action of G_k on $\mathcal{F}(k_s)$. We show that G_k acts continuously on $\mathcal{F}(k_s)$. Consider the stabilizer of any element. Consider $[a]$ for $a \in \mathcal{F}(L)$, L/k finite separable. Then $\operatorname{Stab}_{G_k}(a) = \ker(G \to \operatorname{Gal}(L/k))$ which is open. Therefore the action of G_k is continuous with respect to the discrete topology on $\mathcal{F}(L)$.

   Now we define an inverse map. Let S be a G_k-set. For each finite separable extension L/k contained in k_s, define $\mathcal{F}(L) = S^{\operatorname{Gal}(k_s/L)}$. Since every étale k-scheme U is a disjoint union of k-schemes of the form $\operatorname{Spec} L$, we define $\mathcal{F}(U)$ to be the product of the corresponding $\mathcal{F}(L)$. The restriction maps for L'/L a finite separable extension, where L'/k, L/k are separable finite extensions is given by $S^{\operatorname{Gal}(k_s/L')} \hookrightarrow S^{\operatorname{Gal}(k_s/L)}$. The fact that this defines a sheaf follows by using the axioms from from above. Then $\mathcal{F}(k) = S^{\operatorname{Gal}(k_s/k)} = S^{G_k}$.

2. Clear.

3. The global sections functor corresponds to the G_k-invariants functor. The result follows by taking derived functors.

□

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