Rank Predictions in Elliptic Curves: Parity vs. L-Value Approaches

Warwick Junior Number Theory Seminar

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Plan

1. Parity methods
2. L-value methods
3. Comparison

Points of infinite order

Let E be an elliptic curve over a number field K.

Theorem (Mordell-Weil)

$$E(K) \simeq \mathbb{Z}^r \oplus E(K)_{\text{tors}},$$

where $r = \operatorname{rk} E/K$ *is the* rank *of* E/K.

How to determine $\operatorname{rk} E/K$?

1. Calculate lower bound by finding points of infinite order, e.g. Heegner points for $E(\mathbb{Q})$,
2. Calculate upper bounds using Selmer groups.

⚠ These won't always coincide.

⚠ No effective algorithm for (1).

Parity methods

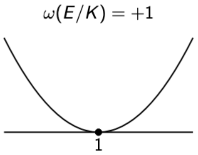
Parity methods

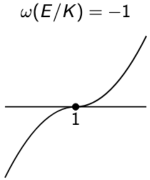
Birch–Swinnerton-Dyer

$\mathrm{rk}\, E/K = \mathrm{ord}_{s=1} L(E/K, s).$

Functional equation

$L^*(E/K, 2-s) = w(E/K) \cdot L^*(E/K, s).$





Parity Conjecture

$$(-1)^{\mathrm{rk}\, E/K} = w(E/K), \quad w(E/K) = \pm 1.$$

Definition (Global root number)

For E/K, the *global root number* is given by

$$w(E/K) = \prod_{v \text{ place of } K} w(E/K_v),$$

where $w(E/K_v)$ are *local root numbers*.

Root number

$$w(E/K) = \prod_{v \text{ place of } K} w(E/K_v).$$

Proposition

Let E/K be a semistable elliptic curve, v a place of K. Then

$$w(E/K_v) = \begin{cases} +1 & E/K_v \text{ good reduction,} \\ +1 & E/K_v \text{ non-split multiplicative reduction,} \\ -1 & E/K_v \text{ split multiplicative reduction,} \\ -1 & v \text{ Archimedean.} \end{cases}$$

Example

Let $E/\mathbb{Q}$ be given by

$$E\colon y^2 + y = x^3 - x.$$

E has non-split multiplicative reduction at $p = 37$. Thus

$$w(E/\mathbb{Q}) = w(E/\mathbb{R}) \cdot w(E/\mathbb{Q}_{37}) = -1 \cdot 1 = -1,$$

and so $\operatorname{rk} E/\mathbb{Q}$ is odd and in particular > 0.

Every $E/\mathbb{Q}$ has even rank over $\mathbb{Q}(\sqrt{-1}, \sqrt{17})$

Let $K = \mathbb{Q}(\sqrt{-1}, \sqrt{17})$. Then every place of $\mathbb{Q}$ splits into an even number of places (2 or 4) in K.

- K has two complex places $v \mid \infty$,
- 2 splits in $\mathbb{Q}(\sqrt{17})$, hence in K also,
- 17 splits in $\mathbb{Q}(\sqrt{-1})$, hence in K also,
- $p \neq 2, 17$ is unramified in K, hence has cyclic decomposition group $\neq C_2 \times C_2$ and so splits also.

For $E/\mathbb{Q}$ an elliptic curve, v, v' places in K above a prime p, one has $w(E/K_v) = w(E/K_{v'})$. Thus

$$w(E/K) = \prod_v w(E/K_v) = \prod (\pm 1)^{\text{even}} = 1,$$

so that E/K has even rank by the Parity Conjecture.

An elliptic curve with infinitely many solutions over $\mathbb{Q}(\sqrt[3]{m})$ for all $m > 1$

Consider

$$E\colon y^2 + y = x^3 + x^2 + x.$$

This has split multiplicative reduction at $p = 19$ and good reduction elsewhere. Let $K = \mathbb{Q}(\sqrt[3]{m})$ for $m > 1$. One has

$$w(E/K) = (-1)^2 \cdot \begin{cases} (-1)^3 & \text{19 splits in } K \\ (-1) & \text{otherwise} \end{cases} = -1.$$

Therefore the Parity Conjecture predicts that E has positive rank over $\mathbb{Q}(\sqrt[3]{m})$ for all $m > 1$ such that $[K\colon \mathbb{Q}] = 3$.

L-value methods

Twisted L-functions and Artin Formalism

$$\text{Elliptic curve } E/K \rightsquigarrow L\text{-function } L(E/K,s)$$

with Euler product

$$L(E/K,s) = \prod_{\mathfrak{p}\subset O_K} \det(1 - N(\mathfrak{p})^{-s}\,\mathrm{Frob}_{\mathfrak{p}}^{-1} \mid (V_l E^*)^{I_{\mathfrak{p}}})^{-1}.$$

Let $G = \mathrm{Gal}(F/K)$ and τ a representation of G.

$$E/K \text{ and } \tau \rightsquigarrow \text{twisted } L\text{-function } L(E/K,\tau,s)$$

with Euler product

$$L(E/K,\tau,s) = \prod_{\mathfrak{p}\subset O_K} \det(1 - N(\mathfrak{p})^{-s}\,\mathrm{Frob}_{\mathfrak{p}}^{-1} \mid (\tau\otimes V_l E^*)^{I_{\mathfrak{p}}})^{-1}.$$

These L-functions satisfy the following properties, known as *Artin formalism*

- $L(E/K,\mathbb{1},s) = L(E/K,s)$,
- $L(E/K,\tau\oplus\tau',s) = L(E/K,\tau,s)\cdot L(E/K,\tau',s)$,
- $L(E/K,\mathbb{C}[G/H],s) = L(E/F^H,s)$ for $H \le G$, $\quad \big(\mathbb{C}[G/H] = \mathrm{Ind}_H^G\,\mathbb{1}\big)$.

Thus

$$\mathbb{C}[G/H] \simeq \bigoplus_i \tau_i^{\oplus n_i} \implies L(E/F^H,s) = \prod_i L(E/K,\tau_i,s)^{n_i}.$$

BSD for twists

If $G = \mathrm{Gal}(F/K)$ and E/K is an elliptic curve, G acts on $E(F)$ by acting coordinate-wise on a point. Since $E(F)$ is a $\mathbb{Z}$-module, $E(F) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a rational representation of G.

Conjecture (BSD Analogue for twists, Deligne-Gross)

Let E be an elliptic curve over a number field K, $G = \mathrm{Gal}(F/K)$, and τ a representation of G. Then

$$\langle \ \tau, \ E(F) \otimes_{\mathbb{Z}} \mathbb{C} \ \rangle = \mathrm{ord}_{s=1} L(E/K, \tau, s).$$

Conjecture (Parity conjecture for twists)

If E is an elliptic curve over a number field K, $G = \mathrm{Gal}(F/K)$ and τ a self-dual representation of G, then

$$(-1)^{\langle \tau, E(F) \otimes_{\mathbb{Z}} \mathbb{C} \rangle} = w(E/K, \tau),$$

where $w(E/K, \tau) = \pm 1$ is the global twisted root number *of E by τ.*

BSD 2

Conjecture (BSD 2)

Let E be an elliptic curve over a number field K. The group $Ш_{E/K}$ has finite order and the leading term of the Taylor series at $s=1$ of the L-function is

$$\lim_{s\to 1}\frac{L(E/K,s)}{(s-1)^r}\cdot\frac{\sqrt{|\Delta_K|}}{\Omega_+(E)^{r_1+r_2}|\Omega_-(E)|^{r_2}}=\underbrace{\frac{\mathrm{Reg}_{E/K}|Ш_{E/K}|C_{E/K}}{|E(K)_{\mathrm{tors}}|^2}}_{:=\mathrm{BSD}(E/K)}$$

If E/K is semistable

$$C_{E/K}=\prod_{\mathfrak{p}\subset O_K}c_{\mathfrak{p}}(E/K),$$

where $c_{\mathfrak{p}}(E/K)=[E(K_{\mathfrak{p}}):E_0(K_{\mathfrak{p}})]$ is the local Tamagawa number. Else, there are some extra factors at the primes of additive reduction.

Question: Can we factor $\mathrm{BSD}(E/K)$ according to Artin representations?

BSD analogue for twists

Conjecture (BSD-term conjecture, [Dokchitser-Evans-Wiersema 21])

*Let $E/\mathbb{Q}$ be an elliptic curve **over the rationals**, and $G = \mathrm{Gal}(F/\mathbb{Q})$. For each representation τ of G there exists an invariant $\mathrm{BSD}(E,\tau) \in \mathbb{C}^\times$ such that:*

1. $\mathrm{BSD}(E,\tau \oplus \tau') = \mathrm{BSD}(E,\tau)\,\mathrm{BSD}(E,\tau')$, *where τ' is a rep. of G,*
2. $\mathrm{BSD}(E,\mathbb{C}[G/H]) = \mathrm{BSD}(E/F^H)$ *for $H \leq G$.*

Assume in addition that $\langle \tau, E(F) \otimes_{\mathbb{Z}} \mathbb{C} \rangle = 0$, and let $\mathbb{Q}(\tau)$ denote the finite abelian extension of $\mathbb{Q}$ obtained by adjoining the values of the character of τ. Then

3. *Galois equivariance:* $\mathrm{BSD}(E,\tau) \in \mathbb{Q}(\tau)^\times$ *and*

$$\mathrm{BSD}(E,\tau^{\mathfrak{g}}) = \mathrm{BSD}(E,\tau)^{\mathfrak{g}} \text{ for } \mathfrak{g} \in \mathrm{Gal}(\mathbb{Q}(\tau)/\mathbb{Q}).$$

Thus

$$\mathbb{C}[G/H] \simeq \bigoplus_i \tau_i^{\oplus n_i} \implies \mathrm{BSD}(E/F^H) = \prod_i \mathrm{BSD}(E,\tau_i)^{n_i}.$$

D_{10} example

Consider $G = \mathrm{Gal}(F/\mathbb{Q}) = D_{10} = \langle x, y \mid y^2 = x^5 = e, yxy = x^{-1} \rangle$.

	e	x	yx	yx^2
$\mathbb{1}$	1	1	1	1
ε	1	-1	1	1
τ	2	0	$\frac{-1-\sqrt{5}}{2}$	$\frac{-1+\sqrt{5}}{2}$
τ^σ	2	0	$\frac{-1+\sqrt{5}}{2}$	$\frac{-1-\sqrt{5}}{2}$

Table: Character table of G with $\mathrm{Gal}(\mathbb{Q}(\sqrt{5})/\mathbb{Q}) = \langle \sigma \rangle$.

One has $\mathbb{C}[G/C_2] \simeq \mathbb{C}[G/G] \oplus \tau \oplus \tau^\sigma$. Let $L = F^{C_2}$, E an elliptic curve over $\mathbb{Q}$ such that $\langle E(F) \otimes_{\mathbb{Z}} \mathbb{C}, \tau \rangle = 0$.

Assuming the BSD-term conjecture,

$$\mathrm{BSD}(E, \mathbb{C}[G/C_2]) = \mathrm{BSD}(E, \mathbb{C}[G/G]) \cdot \mathrm{BSD}(E, \tau) \cdot \mathrm{BSD}(E/\tau^\sigma)$$

i.e.

$$\frac{\mathrm{BSD}(E/L)}{\mathrm{BSD}(E/\mathbb{Q})} = N_{\mathbb{Q}(\sqrt{5})/\mathbb{Q}}(\mathrm{BSD}(E, \tau)),$$

hence is of the form $x^2 - 5y^2$ for x, $y \in \mathbb{Q}$.

D_{10} example

Suppose $\mathrm{rk}\, E/F = 0$. Then

$$\frac{\mathrm{BSD}(E/L)}{\mathrm{BSD}(E/\mathbb{Q})} = \frac{|E(\mathbb{Q})_{\mathrm{tors}}|^2 |\text{Ш}_{E/L}| C_{E/L}}{|E(L)_{\mathrm{tors}}|^2 |\text{Ш}_{E/\mathbb{Q}}| C_{E/\mathbb{Q}}} \equiv \frac{C_{E/L}}{C_{E/\mathbb{Q}}} \pmod{\mathbb{Q}^{\times 2}}$$

Suppose E has semistable reduction over $\mathbb{Q}$. We have $\frac{C_{E/L}}{C_{E/\mathbb{Q}}} = \prod_p \frac{\prod_{w|p} c_w(E/L)}{c_p(E/\mathbb{Q})}$.

- If $E/\mathbb{Q}_p$ has good reduction, $\frac{\prod_{w|p} c_w(E/L)}{c_p(E/\mathbb{Q})} = 1$.
- If $E/\mathbb{Q}_p$ has split multiplicative reduction with $c_p(E/\mathbb{Q}) = n$,

$$\frac{\prod_{w|p} c_w(E/L)}{c_p(E/\mathbb{Q})} = \frac{\prod_{w|p} e_{w/p} \cdot n}{n}.$$

  - If p is inert with ram. deg. 5 in F, this is $5 = 5^2 - 5(2)^2$,
  - If p has decomp. group C_5 and is unramified, this is 1,
  - If p has decomp. group = inertia group = C_2, this is $(2n)^2$.

In all cases (including additive reduction) we have $\frac{C_{E/L}}{C_{E/\mathbb{Q}}} = x^2 - 5y^2$ for some $x, y \in \mathbb{Q}^{\times 2}$ as expected.

D_{24} example

Let

$$f = x^{12} + 22x^8 + 209x^4 + 44$$

and F be the Galois closure of $\mathbb{Q}[x]/(f)$. Then $G = \mathrm{Gal}(F/\mathbb{Q}) = D_{24}$. This field has 12 complex places, and $p = 11$ has residue degree 2 and ramification degree 12.

Let $E/\mathbb{Q}$ be an elliptic curve with additive reduction (of Type II) at $p = 11$ and good reduction elsewhere. Assume that $\mathrm{rk}\, E/F = 0$.

Let τ, τ^σ be the two Galois conjugate 2-dim. irreps. of G with $\mathbb{Q}(\tau) = \mathbb{Q}(\sqrt{3})$. One finds

$$\mathbb{C}[G/C_2] \oplus \mathbb{C}[G/D_6] \simeq \mathbb{C}[G/C_2 \times C_2] \oplus \mathbb{C}[G/S_3] \oplus \tau \oplus \tau^\sigma.$$

The BSD-term conjecture implies

$$\frac{\mathrm{BSD}(E/F^{C_2})\,\mathrm{BSD}(E/F^{D_6})}{\mathrm{BSD}(E/F^{C_2 \times C_2})\,\mathrm{BSD}(E/F^{S_3})} = x^2 - 3y^2, \quad x, y \in \mathbb{Q}.$$

This product is $11 \cdot \square$, which is **not** a norm from $\mathbb{Q}(\sqrt{3})$.

Thus our assumption that $\mathrm{rk}\, E/F = 0$ is false and so $\mathrm{rk}\, E/F > 0$.

Norm Relations Test

Theorem (Norm Relations Test, Dokchitser-Evans-Wiersema)

Suppose the BSD-term conjecture holds. Consider $E/\mathbb{Q}$*, and* $F/\mathbb{Q}$ *with* $G = \mathrm{Gal}(F/\mathbb{Q})$*. Let* τ *be a rep. of* G *with* $\mathbb{Q}(\tau) = \mathbb{Q}(\sqrt{D})$ *and*

$$\bigoplus_j \mathbb{C}[G/H'_j] \simeq \bigoplus_i \mathbb{C}[G/H_i] \oplus \tau \oplus \tau^\sigma$$

for some $H_i, H'_j \leq G$*, where* $\mathrm{Gal}(\mathbb{Q}(\sqrt{D})/\mathbb{Q}) = \langle \sigma \rangle$*. If*

$$\frac{\prod_i C_{E/F^{H_i}}}{\prod_j C_{E/F^{H'_j}}} \notin N_{\mathbb{Q}(\sqrt{D})/\mathbb{Q}}(\mathbb{Q}(\sqrt{D})^\times)$$

then $\mathrm{rk}\, E/F > 0$.

Comparison

D_{24} example

Let

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and F be the Galois closure of $\mathbb{Q}[x]/(f)$. Then $G = \mathrm{Gal}(F/\mathbb{Q}) = D_{24}$. This field has 12 complex places, and $p = 11$ has residue degree 2 and ramification degree 12.

Let $E/\mathbb{Q}$ be an elliptic curve with additive reduction (of Type II) at $p = 11$ and good reduction elsewhere. Assume that $\mathrm{rk}\, E/F = 0$.

Root numbers:

$$w(E/\mathbb{Q}) = w(E/\mathbb{R})\, w(E/\mathbb{Q}_{11}) = (-1) \cdot (-1) = 1,$$

so the PC does not imply that $\mathrm{rk}\, E/\mathbb{Q} > 0$.

Let $L = F^{C_2 \times C_2}$, a non-Galois degree 6 extension. This has 3 complex places and there is one prime above 11 with ramification degree 6. One has

$$w(E/L) = (-1)^3 \cdot (1) = -1,$$

so the PC implies that $\mathrm{rk}\, E/L > 0$ and hence $\mathrm{rk}\, E/F > 0$.

Parity vs. Norm relations test

Question: Does there exist a Galois extension $F/\mathbb{Q}$ and elliptic curve $E/\mathbb{Q}$ such that the Norm relations test predicts $\mathrm{rk}\, E/F > 0$ but Parity methods do not?

Unfortunately . . . no!

Adding in regulators

Theorem

Let $E/\mathbb{Q}$ be an elliptic curve, $G = \mathrm{Gal}(F/\mathbb{Q})$. Suppose that the parity conjecture for twists holds for all self-dual representations τ of G.
Let τ be a rep. of G with $\mathbb{Q}(\tau) = \mathbb{Q}(\sqrt{D})$ and

$$\bigoplus_j \mathbb{C}[G/H'_j] \simeq \bigoplus_i \mathbb{C}[G/H_i] \oplus \tau \oplus \tau^\sigma$$

for some $H_i, H'_j \leq G$, where $\mathrm{Gal}(\mathbb{Q}(\sqrt{D})/\mathbb{Q}) = \langle \sigma \rangle$. If $\langle \tau, E(F)_{\mathbb{C}} \rangle = 0$, then

$$\frac{\prod_i C_{E/F^{H_i}}}{\prod_j C_{E/F^{H'_j}}} \equiv \frac{\prod_i \mathrm{Reg}_{E/F^{H_i}}}{\prod_j \mathrm{Reg}_{E/F^{H'_j}}} \mod N_{\mathbb{Q}(\sqrt{D})/\mathbb{Q}}(\mathbb{Q}(\sqrt{D})^\times).$$

Norms relations test $\implies$ parity

Suppose that the Norms Relations test implies $\mathrm{rk}\, E/F > 0$, i.e. that

$$\frac{\prod_i C_{E/F^{H_i}}}{\prod_j C_{E/F^{H'_j}}} \notin N_{\mathbb{Q}(\sqrt{D})/\mathbb{Q}}(\mathbb{Q}(\sqrt{D})^\times) \implies \frac{\prod_i \mathrm{Reg}_{E/F^{H_i}}}{\prod_j \mathrm{Reg}_{E/F^{H'_j}}} \notin N_{\mathbb{Q}(\sqrt{D})/\mathbb{Q}}(\mathbb{Q}(\sqrt{D})^\times).$$

I can write

$$\frac{\prod_i \mathrm{Reg}_{E/F^{H_i}}}{\prod_j \mathrm{Reg}_{E/F^{H'_j}}} \equiv \prod_\tau C(\tau)^{\frac{1}{k_\tau}\cdot\langle \tau, E(F)_{\mathbb{C}}\rangle} \mod N_{\mathbb{Q}(\sqrt{D})/\mathbb{Q}}(\mathbb{Q}(\sqrt{D})^\times)$$

where τ ranges over the rational irreducible representations of G and $C(\tau) \in \mathbb{Q}^\times$.
Thus there exists some τ such that $\langle \tau, E(F)_{\mathbb{C}}\rangle > 0$. From this one can deduce that there is an intermediate field L such that $w(E/L) = -1$, so the Parity Conjecture implies $\mathrm{rk}\, E/F > 0$ also.

Final comments

- When $G = \mathrm{Gal}(F/\mathbb{Q})$ is a cyclic group or a group of odd order, then $C(\tau) \in \mathbb{Q}^{\times 2}$ for each rational irreducible representation of G. Thus in this case one cannot use the Norms Relations Test to predict that $\mathrm{rk}\, E/F > 0$.
- The proof of the previous theorem uses local root number formulae due to Rohrlich. It suggests that there's a connection between root numbers and Tamagawa numbers, but we don't have a satisfactory explanation for this.
- Generalizations: Higher-dimensional abelian varieties, hyperelliptic curves.

The End

Thank you!