EXPLICIT REGULAR MODELS OF CURVES.

Today's plan : Understand regular models + special fibres of hyperelliptic curves via cluster pictures.

We know that for "nice curves" $\rightsquigarrow$ fibred surface $\overset{\text{desingularize}}{\rightsquigarrow}$ regular model

E.g. if semistable, contract -1 curves $\rightsquigarrow$ min. regular model.

E.g. "nice" $\Rightarrow$ curve attains semistable redn over finite ext.

Setup

(a lot of things work if K is any complete discretely valued field w/ perfect residue field)

* K local field of odd residue char. p

Valuation v, $\mathcal{O}_K$, k res. field (unif. π).

* C/K hyperelliptic curve given by

$$y^2 = f(x) = c\prod_{r\in\mathcal{R}}(x-r)$$

$f\in K[x]$ separable, $\deg(f) = 2g+1$ or $2g+2$ $(g\geq 2)$

$\mathcal{R}$ = roots of $f(x)$.

i.e. the smooth proj. curve assoc. w/ this eqn

= Gluing of pair of affine patches

$U_X : Y^2 = f(X)$ $\qquad$ $U_T : V^2 = T^{2g+2} f(1/T)$

along $X = 1/T$ and $Y = V/T^{g+1}$

§1 Cluster pictures

Definition: Let C/K be a hyperelliptic curve w/ equation

$$y^2 = \underbrace{c_f \prod_{x \in \mathcal{R}} (x - r_i)}_{= f(x)}$$

- A **cluster** is a non-empty subset $\mathfrak{s} \subseteq \mathcal{R}$ of the form $\mathfrak{s} = D \cap \mathcal{R}$ for some disc $D = D_{z_D, d_D} = \{ x \in \overline{K} : v(x - z_D) \geq d_D \}$

  $(z_D \in \overline{K}$ centre , $d_D \in \mathbb{Q}$ radius$)$

- For a cluster $\mathfrak{s}$, w/ $|\mathfrak{s}| > 1$, its **depth** $d_{\mathfrak{s}}$ is the max'l d for which $\mathfrak{s}$ is cut out by such a disc
  i.e. $d_{\mathfrak{s}} = \min_{r, r' \in \mathfrak{s}} v(r - r')$.

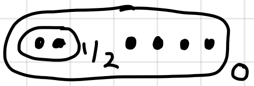
  Specifying containment of clusters $\rightsquigarrow$ cluster picture

Ex: $C: y^2 = x^6 - p \quad \mathcal{R} = \{ \pm p^{1/6}, \pm \zeta_3 p^{1/6}, \pm \zeta_3^2 p^{1/6} \}$

Cluster : $\mathcal{R}$ 

roots equidistant. Good redn in deg 6 ramif. ext.

Ex: $C: y^2 = (x^2 - p)(x^4 + 1)$ Does not have potentially good redn.



(this has semistable redn)

Clusters : $\{ \mathfrak{s} = \{ \pm \sqrt{p} \} , \mathcal{R} \}$

N/B: $G_K \circlearrowright$ clusters via action of G_K on $\mathcal{R}$.

Action preserves depth and containment of clusters.

Def: • $\mathfrak{s}'$ **child** of $\mathfrak{s}$ if $\mathfrak{s}'$ is a max'l subcluster of $\mathfrak{s}$

• $P(\mathfrak{s})$ **parent** of $\mathfrak{s}$ is smallest s.t $\mathfrak{s} \subsetneq P(\mathfrak{s})$

Simplifying assumption #1:

Each cluster $\neq \mathcal{R}$ has size $< 2g$

(slightly more complicated definition w/ out simplifying assumptions)

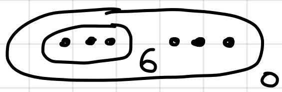
Definition: A cluster $|\mathfrak{s}|$ is **principal** if
- $\mathfrak{s} = \mathcal{R}$: $\mathcal{R}$ has ≥ 3 children
- $\mathfrak{s} \neq \mathcal{R}$: $|\mathfrak{s}| \geq 3$ (not a twin)

* These are the clusters that contribute components of genus ≥ 1 to special fibre

§2 Cluster picture determines special fibre

Idea: Cluster picture tells you how to change the equation of your curve to see different components of the special fibre of a regular model of C/K^{nr} (so special fibre over $\bar{k}$)

Ex: $C: y^2 = (x-p^6)(x-2p^6)(x-3p^6)(x+1)(x+2)(x+3)$



depth of cluster determined chain of $\mathbb{P}^1$s

Claim. This is semistable

D(0,0) $\bar{C}: \quad y^2 = x^3(x+1)(x+2)(x+3)$

principal cluster $\mathcal{R}$: $\bar{C}^\nu: \quad y^2 = x(x+1)(x+2)(x+3) \rightsquigarrow$ genus 1 curve on special fibre of C $\quad \Gamma_1$

D(0,6)

cluster of size 3: $x = p^6 x' \qquad y = p^9 y'$

$C': \cancel{p^{18}} y'^2 = \cancel{p^{18}} (x'-1)(x'-2)(x'-3)(p^6x'+1)(px'+2)(px'+3)$

$\bar{C}': \quad y'^2 = 6(x'-1)(x'-2)(x'-3) \rightsquigarrow$ genus 1 curve on special fibre $\quad \Gamma_2$

See linking chains by other discs:

D(0,2) $\quad x = p^2x' \qquad y = p^3 y'$

$C'': \quad p^6 y'^2 = p^6(x'-p^4)(x'-2p^4)(x'-3p^4)\ldots$

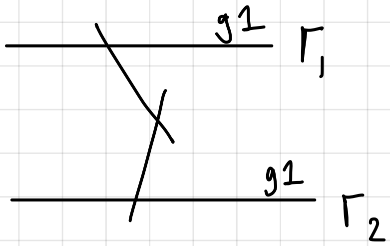
$\bar{C}'': \quad y'^2 = (x')^3 \cdot 6$

$\bar{C}''^\nu: \quad y'^2 = 6x' \quad \rightsquigarrow \quad$ get a $\mathbb{P}^1$.

D(0,1)
D(0,3)
D(0,5)
$y^2 = 0 \ldots$

D(0,4): another $\mathbb{P}^1$.

$\rightsquigarrow$ special fibre



(No -1 curves $\rightsquigarrow$ special fibre of min. reg. model)

$p_\Gamma - g_\Gamma = \#\text{nodes}$

$2g-2 = \sum_\Gamma m_\Gamma (2\cdot p_\Gamma - 2 - (\Gamma\cdot\Gamma))$

Semistability criterion:

ALSO TELLS YOU DEG OF EXT TO ATTAIN S-S RED

C/K is semistable $\iff$

(1) $K(\mathcal{R})/K$ has ramif deg ≤ 2

(2) Every cluster $\mathfrak{s}$ with $|\mathfrak{s}|>1$ is invariant under the action of I_K

(3) Every principal cluster has $d_{\mathfrak{s}} \in \mathbb{Z}$ and

$$\nu_{\mathfrak{s}} = v(c_f) + \sum_{r \in R} \underbrace{d_{\{r\} \wedge \mathfrak{s}}}_{\text{depth of smallest cluster containing } \{r\} \text{ and } \mathfrak{s}} \in 2\mathbb{Z}$$

Thm: Let C/K be semistable.
The special fibre $\mathcal{C}_{min,\bar{k}}$ of the minimal reg. model of C/K^{nr} is determined by the cluster pic
One has

① If $\mathfrak{s}$ is principal and übereven ($= |\mathfrak{s}|$ is even and has all even children) then there are two coresp. irred genus 0 components $\Gamma_{\mathfrak{s}}^+, \Gamma_{\mathfrak{s}}^-$

② For $\mathfrak{s}$ principal and non-übereven, there is one irred component $\Gamma_{\mathfrak{s}}$ of genus $\left\lfloor \frac{\#\text{odd children of } \mathfrak{s} - 1}{2} \right\rfloor$

③ principal $\mathfrak{s}' \lneq \mathfrak{s}$ $\rightsquigarrow$ chain of $\mathbb{P}^1$s from $\Gamma_{\mathfrak{s}}$ to $\Gamma_{\mathfrak{s}'}$ (1 or 2 chains, length $\leftrightarrow$ depth)

Chain of $\mathbb{P}^1$s from $\Gamma_{\mathfrak{s}}$ to $\Gamma_{\mathfrak{s}}$ for each $\mathfrak{t} \leq \mathfrak{s}$, $|\mathfrak{t}|=2$.

④ $\Gamma_{\mathfrak{s}}$ $\mathfrak{s}$ not übereven, Fröbenius acts on $\Gamma_{\mathfrak{s}}$ by

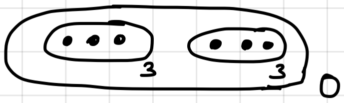
$$\Gamma_{\mathfrak{s}} \longrightarrow \Gamma_{Frob(\mathfrak{s})}$$

(Can also describe action of Frob on $\Gamma_{\mathfrak{s}}^{\pm}$ and $\mathbb{P}^1$ chains)

Example. [Curve with no rational points]

$$C : y^2 = p\cdot((x-i)^3 - p^9)((x+i)^3 - p^9) \quad / \ \mathbb{Q}_p$$

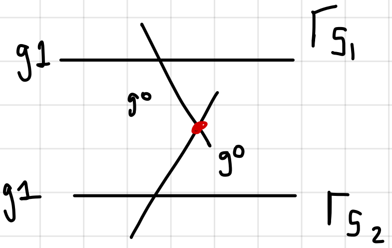
$$\mathcal{R} = \{ p^3 \pm i,\ \zeta_3 p^3 \pm i,\ \zeta_3^2 p^3 \pm i \}$$



$\mathfrak{s}_1 = \{ \zeta_3^j p + i \}$ $\quad \mathfrak{s}_2 = \{ \zeta_3^j p - i \}$

$x = p^3(x' - i) \quad y = p^2 y' \rightsquigarrow$ cluster pic :

$\therefore$ Special fibre of minimal regular model of C/K^{nr} looks like



Frob swaps $\mathfrak{s}_1$ and $\mathfrak{s}_2$

$\rightsquigarrow$ Frob swaps $\Gamma_{\mathfrak{s}_1}$ and $\Gamma_{\mathfrak{s}_2}$ + $\mathbb{P}^1$ components in tail.

$\therefore$ no smooth k-points on special fibre

$\Longrightarrow$ no points on C over $\mathbb{Q}_p$.

$$\left(\begin{array}{l} C(\mathbb{Q}_p^{nr}) = \mathcal{C}_{min}(\mathbb{Z}_p^{nr}) \xrightarrow{red} \overline{\mathcal{C}}_{ns}(\bar{k}) \\ \overline{\mathcal{C}}_{ns}(\bar{k}) \subseteq \overline{\mathcal{C}}(\bar{k}) \text{ non-sing. locus } \ldots \text{ map is surj.} \\ \qquad\qquad\qquad\qquad\qquad \text{by Hensel's lemma} \end{array} \right)$$

§3 (Minimal) regular model of hyperelliptic curve (over K^{nr})

C s.s.

Simplifying condition #2. All clusters $\mathfrak{s}$ w/ $|\mathfrak{s}| > 1$ have integral depth.

Integral disc: $D_{z,d} = \{ x \in \bar{K} : v(x - z) \geq d \}$

w/ $d \in \mathbb{Z} \quad z \in K^{nr}$

$D(\mathcal{R})$ = Smallest disc containing $\mathcal{R}$

Given $\mathcal{R}$, an integral disc is valid if $D \subseteq D(\mathcal{R})$
$\#(\mathcal{R} \cap D) \geq 2$

- For $D = D_{z_D, d_D}$, $P(D) = D_{z_D, d_D - 1}$ parent.

let
- $v_D(f) = v(c) + \sum_{r \in \mathcal{R}} \min\{d_D, v(r - z_D)\}$

$w_D(f) \in \{0, 1\}$ parity of $v_D(f)$.

If $D = D(\mathfrak{S})$ is this the thing in the semistability criterion?

① <u>Construction of a regular model $\mathcal{C}^{disc} / \mathcal{O}_{K^{nr}}$</u>

* For each valid disc D, $f_D(x_D) \in \mathcal{O}_{K^{nr}}[x_D]$

$$f_D(x_D) = \pi^{-v_D(f)} \cdot f(\pi^{d_D} x_D + z_D)$$

$$\mathcal{U}_D = \frac{\operatorname{Spec} \mathcal{O}_{K^{nr}}[x_D, y_D]}{(y_D^2 - \pi^{w_D(f)} \cdot f_D(x_D))} \subseteq \operatorname{Spec} \mathcal{O}_{K^{nr}}[x_D, y_D]$$

$\therefore$ subscheme of $\mathbb{A}^2_{\mathcal{O}_{K^{nr}}}$

$\mathcal{U}_D^\circ \subseteq \mathcal{U}_D$ open subscheme formed by removing all the points in the special fibre corresponding to repeated roots of the redn of f_D.

* For $D(\mathcal{R})$ $g_D(t_D) \in \mathcal{O}_{K^{nr}}[t_D]$

$$g_D(t_D) = t_D^{\deg(f)} f_D(1/t_D)$$

Set $W_D \subseteq \mathbb{A}^2_{\mathcal{O}_{K^{nr}}}$ subscheme cut out by

$$\begin{cases} w_D^2 = \pi^{w_D(f)} \cdot g_D(t_D) & \deg(f) \text{ even} \\ w_D^2 = \pi^{w_D(f)} \cdot t_D \cdot g_D(t_D) & \deg(f) \text{ odd} \end{cases}$$

$W_D^\circ \subseteq W_D$ open subscheme formed by removing all the pts. in the special fibre corresp. to repeated roots of the reduction of g_D.

* For each valid disc $D \neq D(\mathcal{R})$,

$$g_D(s_D, t_D) \in \frac{\mathcal{O}_{K^{nr}}[s_D, t_D]}{(s_D t_D - \pi)} \quad \text{poly satisfying}$$

$$g_D(\pi/t_D, t_D) = t_D^{\nu_D(f) - \nu_{P(D)}(f)} f_D(1/t_D)$$

in $K^{nr}(t_D)$

$W_D \subseteq \mathbb{A}^3_{\mathcal{O}_{K^{nr}}}$ cut out by

$$s_D t_D = \pi, \qquad w_D^2 = s_D^{\omega_D(f)} t_D^{\omega_{P(D)}(f)} g_D(s_D, t_D)$$

$W_D^\circ \subseteq W_D$ open subscheme formed by removing pts in special fibre corresp. to repeated roots of redn of g_D.

(proper)

Theorem 7.3. *A regular model $\mathcal{C}^{\mathrm{disc}}$ of C over $\mathcal{O}_{K^{\mathrm{nr}}}$ is given by gluing each W_D° to $\mathcal{U}_D^\circ$ for each valid D, and to $\mathcal{U}_{P(D)}^\circ$ for all valid $D \neq D(\mathcal{R})$ via the identifications*

$$t_D = 1/x_D = \pi/(x_{P(D)} - \pi^{1-d_D}(z_D - z_{P(D)})),$$

$$s_D = \pi x_D = x_{P(D)} - \pi^{1-d_D}(z_D - z_{P(D)}),$$

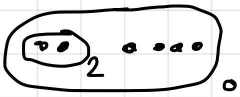
$$w_D = t_D^{\lfloor \nu_D(f)/2 \rfloor - \lfloor \nu_{P(D)}(f))/2 \rfloor} y_D = s_D^{\lfloor \nu_{P(D)}(f))/2 \rfloor - \lfloor \nu_D(f)/2 \rfloor} y_{P(D)}.$$

Remark 7.4. The regular model $\mathcal{C}^{\mathrm{disc}}$ above is not minimal in general: discs with $\omega_D(f) = 1$ produce $\mathbb{P}^1$s in the special fibre with multiplicity 2 and self-intersection -1. Blowing down these components yields the minimal regular model.

* Once we have a regular model, by contracting all -1 curves we get a minimal regular model.
* Then by contracting all -2 curves, get a stable model.

↳ ($\exists$ explicit description of discs s.t. corresp. components need to be contracted)

Example: $C: y^2 = (x^2 - p^4)(x^4+1) \quad / \mathbb{Q}_p$



$\mathfrak{s}, \ |\mathfrak{s}| = 2$

$D_{max} = D(0,0)$
$D' = D(0,1)$
$D'' = D(\mathfrak{s}) = D(0,2)$

$v_D(f) = 0$ $\quad$ $v_{D'}(f) = 2$ $\quad$ $v_{D''}(f) = 4$

$w_D(f) = 0$ $\quad$ $w_{D'}(f) = 0$ $\quad$ $w_{D''}(f) = 0$

- $D_{max} = D$: $\quad f_D(x_D) = f(x_D) \qquad g_D(t_D) = t_D^6 \cdot f(1/t_D)$

$$\mathcal{U}_D = \operatorname{Spec} \frac{\mathbb{Z}_p^{nr}[x_D, y_D]}{(y_D^2 - (x_D^2 - p^4)(x_D{}^4+1))} \qquad \mathcal{W}_D = \operatorname{Spec} \frac{\mathbb{Z}_p^{nr}[w_D, t_D]}{(w_D^2 - (1 - p^4 t_D^2)(1+t_D^4))}$$

$$\mathcal{U}_D^0 = \mathcal{U}_D \setminus \{(p, x_D, y_D)\} \qquad \mathcal{W}_D^0 = \mathcal{W}_D$$

- D': $\quad f_{D'}(x_{D'}) = p^{-2} f(p x_{D'})$

$$\mathcal{U}_{D'} = \operatorname{Spec} \frac{\mathbb{Z}_p^{nr}[x_{D'}, y_{D'}]}{(y_{D'}^2 - (x_{D'}^2 - p^2)(p^4 x_{D'}^4 + 1))} \qquad \mathcal{U}_{D'}^0 = \mathcal{U}_{D'} \setminus \{(p, x_{D'}, y_{D'})\}$$

$$\mathcal{W}_{D'} = \operatorname{Spec} \frac{\mathbb{Z}_p^{nr}[w_{D'}, s_{D'}, t_{D'}]}{(s_{D'} t_{D'} - p, \ w_{D'}^2 - (1 - p^2 t_{D'})(s_{D'}^4 + 1))} \qquad \mathcal{W}_{D'}^0 = \mathcal{W}_{D'}$$

- D'': $\quad f_{D''}(x_{D''}) = p^{-4} f(p^2 x_{D''})$

$$\mathcal{U}_{D''} = \operatorname{Spec} \frac{\mathbb{Z}_p^{nr}[x_{D''}, y_{D''}]}{(y_{D''}^2 - (x^2-1)(p^8 x^4 + 1))} \qquad \mathcal{U}_{D''}^0 = \mathcal{U}_D$$

$$\mathcal{W}_{D''} = \operatorname{Spec} \frac{\mathbb{Z}_p^{nr}[w_{D''}, s_{D''}, t_{D''}]}{(s_{D''} t_{D''} - p, \ w_{D''}^2 - (1 - t_{D''}^2)(p^4 s_{D''} + 1))} \qquad \mathcal{W}_{D''} = \mathcal{W}_D^0$$

- Glue $\mathcal{W}_D^0 \to \mathcal{U}_D^0$ via $t_D = 1/x_D \quad w_D = x_D^{-3} y_D$
- Glue $\mathcal{W}_{D'}^0 \to \mathcal{U}_{D'}^0$ via $t_{D'} = 1/x_{D'} \quad w_{D'} = t_{D'} \cdot y_{D'}$
  $s_{D'} = p x_{D'}$

- Glue $W_{D''}^{0} \to U_{D''}^{0}$ via $t_{D''} = 1/x_{D'}$ $\quad w_{D''} = t_{D''}\, y_{D''}$

$$s_{D''} = p\, x_{D''}$$

- Glue $W_{D'}^{0} \to U_{D}^{0}$ $\quad t_{D'} = p/x_D \quad w_{D'} = s_{D'}^{-1} y_D$

$$s_{D'} = x_D$$

- Glue $W_{D''}^{0} \to U_{D'}^{0}$ $\quad t_{D''} = p/x_{D'} \quad w_{D''} = s_{D''}^{-1} y_{D'}$

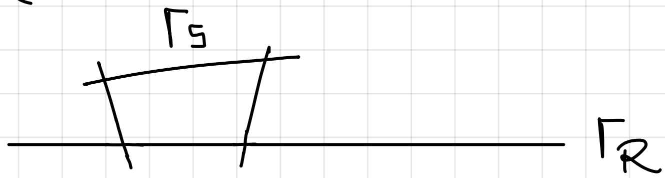
$$s_{D''} = x_{D'}$$

<u>Special fibre</u> : $\begin{cases} y_D^2 = x_D^2 (x_D^4+1) \\ w_D^2 = 1 + t_D^4 \end{cases}$ $\quad$ genus 1

$\left(\dfrac{y_{D'}}{x_{D'}}\right)^2 = 1$

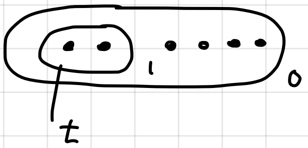
$\begin{cases} y_{D'}^2 = x_{D'}^2 \\ s_{D'} \cdot t_{D'} = 0 \\ w_{D'}^2 = s_{D'}^4 + 1 \end{cases}$ $\quad$ genus 0 $\times 2$

$\begin{cases} y_{D''}^2 = x_{D''}^2 - 1 \\ s_{D''} \cdot t_{D''} = 0 \\ w_{D''}^2 = 1 - t_{D''}^2 \end{cases}$ $\quad$ genus 0



Example: (genus 2 semistable)

$C : y^2 = (x^2 - p^2)(x^4 + 1)$



$D = D_{max} = D(0,0)$

$D' = D(0,1) = D(t)$

$z_D = z_{D'} = 0 \qquad v_D(f) = 0 \qquad v_{D'}(f) = 2$

$$f_D(x) = (x^2 - p^2)(x^4 + 1)$$

$$f_{D'}(x) = p^{-2}(p^2x^2 - p^2)(p^4x^4 + 1) = (x^2 - 1)(p^4x^4 + 1)$$

$$\mathcal{U}_D = \operatorname{Spec} \frac{\mathbb{Z}_p^{nr}[x_D, y_D]}{(y_D^2 - (x_D^2 - p^2)(x_D^4 + 1))} \qquad \mathcal{U}_{D'} = \operatorname{Spec} \frac{\mathbb{Z}_p^{nr}[x_{D'}, y_{D'}]}{(y_{D'}^2 - (x_{D'}^2 - 1)\cdot(p^4 x_{D'}^4 + 1))}$$

$$\mathcal{U}_D^\circ = \mathcal{U}_D \setminus \{(0,0,0)\} \qquad \mathcal{U}_{D'}^\circ = \mathcal{U}_{D'}$$

$$g_D(t) = t^6 \cdot (1/t^2 - p^2)(1/t^4 + 1) = (1 - p^2t^2)(1 + t^4)$$

$$W_D = \operatorname{Spec} \frac{\mathbb{Z}_p^{nr}[w_D, t_D]}{(w_D^2 - (1 - p^2t_D^2)(1 + t_D^4))} \qquad W_{D'}^\circ = W_D$$

$$g_{D'}(p/t, t) = t^2(1/t^2 - 1)(p^4 / t^4 + 1) = (1 - t^2)(\underbrace{(p/t)}_{s}{}^4 + 1)$$

$$\therefore\ g_{D'}(s,t) = (1 - t^2)(s^4 + 1)$$

$$W_{D'} = \operatorname{Spec} \frac{\mathbb{Z}_p^{nr}[w_{D'}, s_{D'}, t_{D'}]}{(s_{D'}t_{D'} - \pi,\ w_{D'}^2 - (1 - t_{D'}^2)(s_{D'}^4 + 1))} \qquad W_{D'}^\circ = W_{D'}.$$

Regular model $\mathcal{C}^{disc}$ of C over $\mathbb{Z}_p^{nr}$ given by gluing

$W_D^\circ \to \mathcal{U}_D^\circ$ via. $t_D = 1/x_D \qquad w_D = x_D^{-3} y_D$

by gluing $W_{D'}^{\circ} \to U_{D'}^{\circ}$ via $t_{D'} = 1/x_{D'}$

$w_{D'} = t_{D'} y_{D'}$

and glue $W_{D'}^{\circ} \to U^{\circ}_{D}$ via

$t_{D'} = p/x_D \qquad s_{D'} = x_D$

$w_{D'} = s_{D'}^{-1} y_D$

look @ special fibre: $\left\{\begin{array}{l} y_D^2 = x_D^2(x_D^4+1) \\ w_D^2 = 1 + t_D^4 \end{array}\right.$ $\setminus \{(0,0)\}$

curve of genus 1

$y_{D'}^2 = (x_{D'}^2 - 1)$

$s_{D'} t_{D'} = 0$

curve of genus 0

$w_{D'}^2 = (1 - t_{D'}^2)(s_{D'}^4 + 1)$

$s_{D'} = 0$: $w_{D'}^2 = 1 - t_{D'}^2$

$t_{D'} = 0$: $w_{D'}^2 = s_{D'}^4 + 1$

genus 0 (rational bridge)

$w_D^2 = 1 + t_D^4$

$w_{D'}^2 = 1 - t_{D'}^2$

$\cap$ at

points @ ∞

$(t_D, w_D) = (0, \pm 1)$

$(t_{D'}, s_{D'}, w_{D'}) = (0, 0, \pm 1)$

To obtain stable model, contract bridge. Special fibre looks like

If $C: y^2 = (x^2 - p^{2n})(x^4+1)$, special fibre of minimal regular model looks like