Reduction types of genus 2 curves

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Art in mathematics

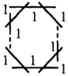
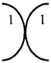
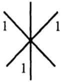
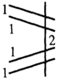
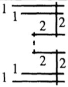
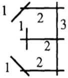
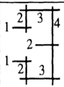
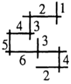
Kodaira symbol	I_0	I_n $(n \geq 1)$	II	III	IV	I_0^*	I_n^* $(n \geq 1)$	IV^*	III^*	II^*
Special fiber $\tilde{\mathcal{C}}$ (The numbers indicate multiplicities)										
$m =$ number of irred. components	1	n	1	2	3	5	$5+n$	7	8	9
$E(K)/E_0(K) \cong \tilde{\mathcal{E}}(k)/\tilde{\mathcal{E}}^0(k)$	(0)	$\frac{\mathbb{Z}}{n\mathbb{Z}}$	(0)	$\frac{\mathbb{Z}}{2\mathbb{Z}}$	$\frac{\mathbb{Z}}{3\mathbb{Z}}$	$\frac{\mathbb{Z}}{2\mathbb{Z}} \times \frac{\mathbb{Z}}{2\mathbb{Z}}$	$\frac{\mathbb{Z}}{2\mathbb{Z}} \times \frac{\mathbb{Z}}{2\mathbb{Z}}$ n even / $\frac{\mathbb{Z}}{4\mathbb{Z}}$ n odd	$\frac{\mathbb{Z}}{3\mathbb{Z}}$	$\frac{\mathbb{Z}}{2\mathbb{Z}}$	(0)
$\tilde{\mathcal{E}}^0(k)$	$\tilde{E}(k)$	k^*	k^+	k^+	k^+	k^+	k^+	k^+	k^+	k^+
Entries below this line only valid for $\mathrm{char}(k) = p$ **as indicated**										
$\mathrm{char}(k) = p$			$p \neq 2,3$	$p \neq 2$	$p \neq 3$	$p \neq 2$	$p \neq 2$	$p \neq 3$	$p \neq 2$	$p \neq 2,3$
$v(\mathcal{D}_{E/K})$ (discriminant)	0	n	2	3	4	6	$6+n$	8	9	10
$f(E/K)$ (conductor)	0	1	2	2	2	2	2	2	2	2
behavior of j	$v(j) \geq 0$	$v(j) = -n$	$\tilde{j} = 0$	$\tilde{j} = 1728$	$\tilde{j} = 0$	$v(j) \geq 0$	$v(j) = -n$	$\tilde{j} = 0$	$\tilde{j} = 1728$	$\tilde{j} = 0$

Table 4.1: A Table of Reduction Types

Genus 2

Hyperelliptic: $y^2 = f(x)$, $\deg f = 6$.

100+ families of reduction types (Namikawa-Ueno labels). Split into 7 semistable types:

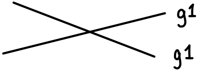
Good (18)

One node (12)

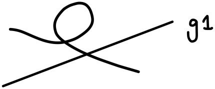
Two nodes (5)

Two $\mathbb{P}^1$s $\cap$ at 3 pts (6)

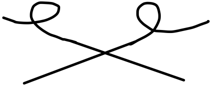
Good EC$\times$Good EC (42)



Mult EC$\times$Good EC (16)



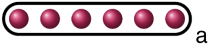
Mult EC$\times$Mult EC (5)



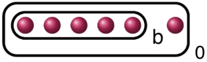
Potentially good reduction

Consider $C: y^2 = c \cdot \prod_{i=1}^{6}(x - r_i)$, $\quad r_i \in \overline{\mathcal{K}}$, $c \in \mathcal{K}^\times$ (res. char $\mathcal{K} > 5$).

If $C/\mathcal{K}$ has potentially good reduction, one can obtain an equation for C with one of the following associated cluster pictures:



$a \in \{0, \frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}\}$



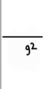
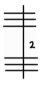
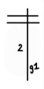
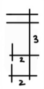
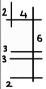
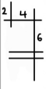
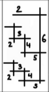
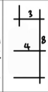
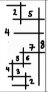
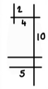
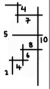
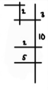
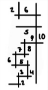
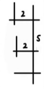
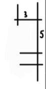
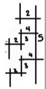
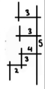
$b \in \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}\}$

(Faraggi-Nowell)

The reduction type is determined by the pair:

$$(\mathrm{a},\ v(c) \bmod 2) \quad \text{or} \quad (\mathrm{b},\ v(c) \bmod 2)$$

For your consideration

Namikawa-Ueno label	I_0	I_0^*	II	III	IV	V	V^*	VI	VII	VII^*	VIII-1	VIII-2	VIII-3	VIII-4	IX-1	IX-2	IX-3	IX-4
Special fibre																		
Number of components	1	7	3	7	6	5	12	7	5	12	5	9	6	13	6	5	11	9
$\Phi(\bar{k})$	(0)	$\left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^4$	(0)	$\left(\frac{\mathbb{Z}}{3\mathbb{Z}}\right)^2$	(0)	$\frac{\mathbb{Z}}{3\mathbb{Z}}$	$\frac{\mathbb{Z}}{3\mathbb{Z}}$	$\left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^2$	$\frac{\mathbb{Z}}{2\mathbb{Z}}$	$\frac{\mathbb{Z}}{2\mathbb{Z}}$	(0)	(0)	(0)	(0)	$\frac{\mathbb{Z}}{5\mathbb{Z}}$	$\frac{\mathbb{Z}}{5\mathbb{Z}}$	$\frac{\mathbb{Z}}{5\mathbb{Z}}$	$\frac{\mathbb{Z}}{5\mathbb{Z}}$
$\mathrm{char}(\bar{k}) \neq 2, 3, 5$																		
 a	0	0	1/2, 1/2	1/3, 2/3	1/3, 2/3	1/6, 5/6	1/6, 5/6				4/5	2/5	3/5	1/5	3/5	1/5	4/5	2/5
 b	0	0						1/2, 1/2	1/4, 3/4	1/4, 3/4	1/5	3/5	2/5	4/5	2/5	4/5	1/5	3/5
f (conductor)	0	4	2	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4
$v(\Delta_{\min})$	0	10	15	10	20	5	15	10	5	15	4	12	18	16	8	6	14	12

A Table of Potentially Good Reduction Types

■ : $v(c) \equiv 0 \mod 2$,
■ : $v(c) \equiv 1 \mod 2$.