Elliptic Curves with positive rank

Elliptic Curves in the Cotswolds meeting

Edwina Aylward

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Not a Brauer relation

Throughout, $K/\mathbb{Q}$ is a quadratic extension.

Definition

Let G be a finite group. A formal sum of subgroups $\Theta = \sum_i n_i H_i$, $n_i \in \mathbb{Z}$ is a *K-relation* if there exists a representation ρ of G with $\mathbb{Q}(\rho) \subset K$ and

$$\bigoplus_i \mathbb{C}[G/H_i]^{\oplus n_i} \simeq \rho \oplus \rho^\sigma,$$

where σ generates $\mathrm{Gal}(K/\mathbb{Q})$.

Example

If $\rho = 0$, Θ is called a *Brauer relation*.

Example

$G = C_p$,

$$\mathbb{C}[G/C_1] \ominus \mathbb{C}[G/G] \simeq \bigoplus_{\tau \in \mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})} \chi^\tau$$

$\implies$ $C_1 - G$ is a $\mathbb{Q}(\sqrt{p^*})$-relation.

Use?

$E/\mathbb{Q}$ elliptic curve, $F/\mathbb{Q}$ finite Galois extension with $G = \mathrm{Gal}(F/\mathbb{Q})$.

$$\bigoplus_i \mathbb{C}[G/H_i]^{\oplus n_i} \simeq \rho \oplus \rho^\sigma \implies \prod_i L(E/F^{H_i}, s)^{n_i} = L(E, \rho, s) \cdot L(E, \rho^\sigma, s).$$

Look at $\mathrm{ord}_{s=1}$. BSD + L-value predictions imply that if $\langle \rho, E(F) \otimes_{\mathbb{Z}} \mathbb{C} \rangle = 0$, then

$$\prod_i (C_{E/F^{H_i}} \cdot \mathrm{Reg}_{E/F^{H_i}})^{n_i} \in N_{K/\mathbb{Q}}(K^\times),$$

where $C_{E/F^{H_i}}$ = product of local Tamagawa numbers + other local fudge factors.

Norm relations test

If

$$\prod_i (C_{E/F^{H_i}})^{n_i} \notin N_{K/\mathbb{Q}}(K^\times),$$

then $\mathrm{rk}\, E/F > 0$.

Norm relations test

Example

Let $F/\mathbb{Q}$ be a finite Galois extension with $G = \mathrm{Gal}(F/\mathbb{Q}) = D_{21}$. For $K = \mathbb{Q}(\sqrt{21})$, have

$$\mathbb{C}[G/C_2] \ominus \mathbb{C}[G/D_7] \ominus \mathbb{C}[G/S_3] \oplus \mathbb{C}[G/G] \simeq \rho \oplus \rho^{\sigma}$$

for a representation ρ of G with $\mathbb{Q}(\rho) = K$ and $\langle\sigma\rangle = \mathrm{Gal}(K/\mathbb{Q})$.
Let $E/\mathbb{Q}$ be a semistable elliptic curve $\rightsquigarrow$ look at

$$\frac{C_{E/F^{C_2}} \cdot C_{E/\mathbb{Q}}}{C_{E/F^{D_7}} \cdot C_{E/F^{S_3}}} \mod N_{K/\mathbb{Q}}(K^{\times}).$$

e.g. If $E/\mathbb{Q}$ has split multiplicative reduction at a prime p with residue degree 2 and ramification degree 3, and good reduction at all other ramified primes in F, then

$$\frac{C_{E/F^{C_2}} \cdot C_{E/\mathbb{Q}}}{C_{E/F^{D_7}} \cdot C_{E/F^{S_3}}} \equiv 3 \mod N_{K/\mathbb{Q}}(K^{\times})$$

$\implies \mathrm{rk}\, E/F > 0.$

Enter parity

Theorem

Let F/L be a finite Galois extension of number fields, $G = \mathrm{Gal}(F/L)$, and $\Theta = \sum_i n_i H_i$ a K-relation. Let E/L be an elliptic curve [1]*. Then*

$$\prod_i (C_{E/F^{H_i}})^{n_i} \equiv \prod_{\tau \in \mathrm{Irr}_{\mathbb{Q}}(G)} \mathcal{C}_\Theta(\tau)^{u(E,\chi_\tau)} \mod N_{K/\mathbb{Q}}(K^\times),$$

- χ_τ *is a* $\mathbb{C}$*-irreducible constituent of* τ*,*
- $u(E,\chi_\tau) \in \{0,1\}$ *satisfies* $w(E,\chi_\tau) = (-1)^{u(E,\chi_\tau)}$*,*
- $\mathcal{C}_\Theta(\tau)$ *is the regulator constant associated to* τ*.*

Corollary

Let $F/\mathbb{Q}$ be a finite Galois extension with $G = \mathrm{Gal}(F/\mathbb{Q})$, and $E/\mathbb{Q}$ an elliptic curve [1]*. If the norm relations test predicts $\mathrm{rk}\, E/F > 0$, then $w(E,\chi) = -1$ for some irreducible representation χ of G.*

Parity conjecture for twists:

$$(-1)^{\langle \chi, E(F)\otimes_{\mathbb{Z}}\mathbb{C}\rangle} = w(E,\chi)$$

~~$\implies \chi$ is a subrep. of $E(F) \otimes_{\mathbb{Z}} \mathbb{C}$.~~

[1] E must have semistable reduction at primes above 2 and 3

Regulator constants

Given a group G and K-relation Θ, one can associate to a rational representation τ of G its regulator constant $\mathcal{C}_\Theta(\tau) \in \mathbb{Q}^\times$. It's value in $\mathbb{Q}^\times / N_{K/\mathbb{Q}}(K^\times)$ is independent of any choices in the definition.
Regulator constants were introduced by Tim and Vladimir for the case where Θ is a Brauer relation. These are purely group-theoretic constants.

Example

Let $G = D_{21}$, with $\mathbb{Q}(\sqrt{21})$-relation $\Theta = C_2 - D_7 - S_3 + G$. The irreducible rational representations of G are $\{\mathbb{1}, \varepsilon, \sigma_3, \sigma_7, \sigma_{21}\}$. The regulator constants are

$$\mathcal{C}_\Theta(\mathbb{1}) \equiv \mathcal{C}_\Theta(\epsilon) \equiv \mathcal{C}_\Theta(\sigma_3) \equiv 1, \quad \mathcal{C}_\Theta(\sigma_7) \equiv \mathcal{C}_\Theta(\sigma_{21}) \equiv 3 \mod N_{\mathbb{Q}(\sqrt{21})/\mathbb{Q}}(\mathbb{Q}(\sqrt{21})^\times).$$

Thus

$$\frac{C_{E/F^{C_2}} \cdot C_{E/\mathbb{Q}}}{C_{E/F^{S_3}} \cdot C_{E/F^{D_7}}} \equiv 3^{u(E,\chi_{21})+u(E,\chi_7)} \mod N_{\mathbb{Q}(\sqrt{21})/\mathbb{Q}}(\mathbb{Q}(\sqrt{21})^\times),$$

where χ_{21} is a faithful $\mathbb{C}$-irreducible representation of G, and χ_7 is the lift of a faithful $\mathbb{C}$-irreducible representation of the D_7-quotient of G.

norm relations test predicts $\operatorname{rk} E/F > 0$ $\implies$ $w(E, \chi_{21}) = -1$ or $w(E, \chi_7) = -1$, (but not both).

The End

Thank you!