WEEK 7

[Reference : § 1.7/1.8 Iwahori - Matsumoto : On some Bruhat decomposition and the structure of the Hecke rings of p-adic Chevalley groups]

Before we talk about plan/aim, lets recap what Megan set up last week.

Recap :

- $\mathfrak{g}_{\mathbb{C}}$ complex s.s lie alg.

- $\mathfrak{g}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_{\alpha}$ Cartan decomposition

  $\mathfrak{h}_{\mathbb{C}}$ ↑ Cartan subalgebra

  Φ → associated root system.

- Let $\Delta = \{\alpha_1, \dots, \alpha_\ell\}$ be a root basis.
  α_0 = highest root.

- $E = \mathfrak{h}_{\mathbb{R}}^* = \mathbb{R}.\Phi$

- $P_{\alpha,k} = \{x \in E^* : \langle \alpha, x \rangle = x(\alpha) = k\}$ $\quad \alpha \in \Phi$, $k \in \mathbb{Z}$

  $\langle -,- \rangle : E \times E^* \longrightarrow \mathbb{C}$

- $w_{\alpha,k} : E^* \longrightarrow E^*$

  $x \longmapsto x - (\langle \alpha, x \rangle - k) \cdot \alpha^\vee$

  $\hookrightarrow \alpha^\vee \in E^*$, $\langle \alpha, \alpha^\vee \rangle = 2$

  Reflection across $P_{\alpha,k}$.

  Write $w_{\alpha,k} = T(k\alpha^\vee) \circ w_\alpha$

  $T(k\alpha^\vee)$ ↓ translation in E^* by $k\alpha^\vee$

  $\hookrightarrow w_\alpha := w_{\alpha,0}$

- Lattice of weights : $P = \{\lambda \in E . \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z} \ \forall \alpha \in \Phi\}$

  Lattice of roots : $Q = \mathbb{Z}\Phi$

  Dual lattices :

  $P^\perp = \{x \in E^* : \langle \lambda, x \rangle \in \mathbb{Z} \ \forall \lambda \in P\}$

  $Q^\perp = \{x \in E^* : \langle \lambda, x \rangle \in \mathbb{Z} \ \forall \lambda \in Q\}$

Then $Q \leq P \qquad P^{\perp} \leq Q^{\perp}$

$[P : Q] < \infty$, order $= |\det(A)|$ A cartan matrix

$=$ order of centre of s.c. lie group w/ lie alg. $\mathfrak{g}_{\mathbb{C}}$.

$Q^{\perp} = \sum_{i=1}^{\ell} \mathbb{Z}\, \varepsilon_i$ w/ $\varepsilon_i \in E^*$

$\varepsilon_i(\alpha_j) = \langle \alpha_j , \varepsilon_i \rangle = \delta_{ij}$

- Let $D = \langle T(\alpha) : \alpha \in Q^{\perp} \rangle$

$D' = \langle T(\alpha) : \alpha \in P^{\perp} \rangle \qquad D' \leq D$

<u>Weyl group</u> : $W = \langle w_\alpha : \alpha \in \Delta \rangle$

<u>Extended weyl group</u> : DW

<u>Affine weyl group</u> : $D'W$

$DW \circlearrowright E^* \setminus \bigcup_{\alpha, k} P_{\alpha, k}$ cells.

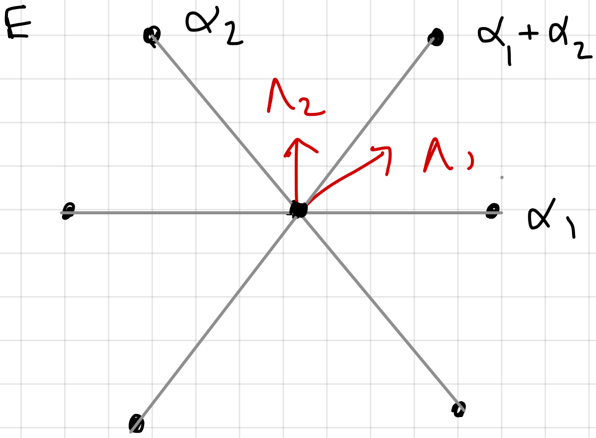
- Fundamental cell $D_0 = \{ x \in E^* : 0 < \langle \alpha, x \rangle < 1 \ \forall \alpha \in \Phi^+ \}$

$\boxed{\text{Todays plan}}$:
- Describe the stabilizer $\Omega \leq DW$ of the fundamental cell in terms of vertices of the fundamental cell.
- Show Ω acts on the affine Dynkin diagram.

Example

$R = A_2$ (simply laced, root system of SL_3).

E



$\alpha_1 + \alpha_2 = \alpha_0$ longest root

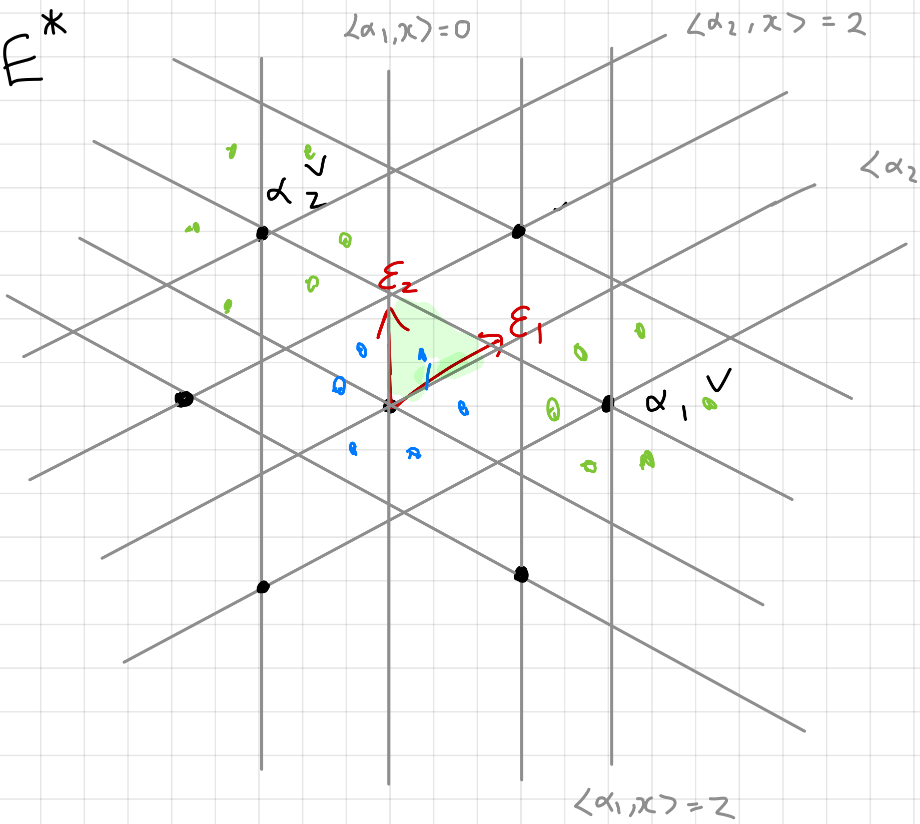
$\left(\begin{array}{l} |\Phi| = 6 \\ \Delta = \{\alpha_1, \alpha_2\} \end{array} \right)$

Fundamental weights:

$$\Lambda_1 = 2/3\, \alpha_1 + 1/3\, \alpha_2$$
$$\Lambda_2 = 1/3\, \alpha_1 + 2/3\, \alpha_2$$

$[P : Q] = 3$.

E^*



$$Q^\perp = \mathbb{Z}\varepsilon_1 \oplus \mathbb{Z}\varepsilon_2$$

$$\varepsilon_1 = \frac{2}{3}\alpha_1^\vee + \frac{1}{3}\alpha_2^\vee$$
$$\varepsilon_2 = 1/3\,\alpha_1^\vee + \frac{2}{3}\alpha_2^\vee$$

$$P^\perp = \mathbb{Z}\alpha_1^\vee \oplus \mathbb{Z}\alpha_2^\vee$$

$\boxed{\S 1.7}$

Recall $D_0 = \{x \in E^* : \langle \alpha_i, x \rangle > 0 \ , \ \langle \alpha_0, x \rangle < 1\}$

<u>Def</u> $\Omega = \{\sigma \in DW : \sigma D_0 = D_0\}$
$= \{\sigma \in DW : \lambda(\sigma) = 0\}$

(Recall $\lambda(\sigma) = |\tilde{\Delta}(\sigma)| = |\{P_{\alpha,x} : D_0 \not\mid \sigma D_0 \mid P_{\alpha,x}\}|$)

Last week Megan proved that $D'W$ acts simply transitively on cells $\Rightarrow D'W \cap \Omega = \{1\}$
$\Rightarrow DW = \Omega \cdot (D'W)$ semi-direct product,

$D'W \trianglelefteq DW$
and $\Omega \simeq DW / D'W \simeq Q^{\perp} / P^{\perp} \simeq P / Q$
is an abelian group of finite order $[P : Q]$

<u>Rmk</u>: If $\sigma \in DW$, $\rho, \rho' \in \Omega$ then
$\lambda(\rho \sigma \rho') = \lambda(\sigma)$ as
$\tilde{\Delta}(\rho\sigma\rho') = \tilde{\Delta}(D_0, \rho\sigma\rho' D_0)$
$= \tilde{\Delta}(D_0, \rho\sigma D_0) = \rho \tilde{\Delta}(D_0, \sigma D_0)$.

$\boxed{\text{Describing } \Omega}$

Consider $\sigma = T(d) \cdot w \in \Omega$ $\quad d \in Q^{\perp}$ $\quad w \in W$.
If $\sigma \neq 1 \Rightarrow d \neq 0$ as $\Omega \cap W \subset D'W \cap \Omega = \{1\}$
$\sigma D_0 = D_0 \Rightarrow \sigma \overline{D_0} = \overline{D_0}$
$\Rightarrow \underbrace{\sigma(0)}_{= d} \in \overline{D_0}$

so $d \in Q^+ \cap \overline{D_0}$.

i.e. $\alpha_0 = \alpha_i + \sum_{j \neq i} k_j \alpha_j$

<u>Claim</u>: $Q^+ \cap \overline{D_0} = \{0\} \cup \{ \varepsilon_i : \langle \alpha_0, \varepsilon_i \rangle = 1 \}$

(Proposition 1.17)

$\therefore \quad \sigma = T(\varepsilon_i) . \omega \quad$ w/ ε_i s.t $\langle \alpha_0, \varepsilon_i \rangle = 1$.

ε_i determines σ uniquely:

If $T(\varepsilon_i)\omega \; , T(\varepsilon_i)\omega' \in \Omega$ then

$$\omega^{-1} T(\varepsilon_i - \varepsilon_i)\omega' = \omega^{-1}\omega' \in \Omega \cap W$$

$$\Rightarrow \omega = \omega'.$$

<u>Conversely</u>, WTP that if $d = \varepsilon_i$ w/ $\langle \alpha_0, \varepsilon_i \rangle = 1$

$\Rightarrow \exists \omega \in W$ s.t $T(d)\omega \in \Omega$ (and ω will be unique)

Recall W acts on {root bases} simply transitively

$\therefore \exists ! \; \omega_\Delta \in W$ w/ $\omega_\Delta(\Delta) = -\Delta$ and
(longest element) $\omega_\Delta{}^2 = 1$

(viewing $\omega_\Delta : E^* \longrightarrow E^*$, $\omega_\Delta(\alpha_i^\vee) = -\alpha_i^\vee$)

(this is a new root system)

For $1 \le i \le l$, let $\Delta_i = \Delta \setminus \{\alpha_i\}$. Then $\exists !$

$\omega_{\Delta_i} \in W_i := \langle \omega_{\alpha_j} : j \neq i \rangle \le W$ s.t

$\omega_{\Delta_i}(\Delta_i) = -\Delta_i \quad , \quad \omega_{\Delta_i}{}^2 = 1$. (longest elem.)

<u>Proposition</u>: $T(\varepsilon_i)\,\omega_{\Delta_i}\,\omega_\Delta \in \Omega$ when $\langle \alpha_0, \varepsilon_i \rangle = 1$

<u>Proof</u>. One has $\omega_\Delta(D_0) = -D_0$. Let $a \in D_0$,

WTP $\varepsilon_i + \omega_{\Delta_i}(b) \in D_0$ where

$b = \omega_\Delta(a)$.

Since ω_{Δ_i} is a combination of ω_{α_j} $j \neq i$,

$$\omega_{\Delta_i}(\alpha_i) = \alpha_i + \sum_{j \neq i} \underbrace{k_j}_{\in \mathbb{Z}_{\geq 0}} \alpha_j$$

consider ω_{Δ_i} acting on roots

since $\langle \alpha_j, \alpha_i \rangle \leq 0$

$\therefore$ $\omega_{\Delta_i}(\alpha_i) > 0$.

Also $\omega_{\Delta_i}(\alpha_0) > 0$

<u>Have</u>:

- $\langle \alpha_j, \varepsilon_i + \omega_{\Delta_i}(b) \rangle = \langle \alpha_j, \omega_{\Delta_i}(b) \rangle$

$$= \langle \omega_{\Delta_i}(\alpha_j), b \rangle$$

$$= b(\underbrace{\omega_{\Delta_i}(\alpha_j)}_{\in -\Delta_i}) > 0 \quad \text{as } b \in D_0$$

- $\langle \alpha_i, \varepsilon_i + \omega_{\Delta_i}(b) \rangle$

$$= 1 + \underbrace{\langle \underset{\in \Phi^+}{\omega_{\Delta_i}(\alpha_i)}, \underset{\in D_0}{b} \rangle}_{-1 < * < 0}$$

$\therefore$ get > 0.

- $\langle \alpha_0, \varepsilon_i + \omega_{\Delta_i}(b) \rangle$

$$= 1 + \underbrace{\langle \underset{\in \Phi^+}{\omega_{\Delta_i}(\alpha_0)}, b \rangle}_{-1 < * < 0}$$

$\therefore$ < 1. $\square$

$\rightsquigarrow$ <u>Prop 118</u>. Have a bijection

$$\{0\} \cup \{\varepsilon_i : \langle \alpha_0, \varepsilon_i \rangle = 1\} \longrightarrow \Omega$$

$$0 \longmapsto id$$

$$\varepsilon_i \longmapsto T(\varepsilon_i) \cdot \omega_{\Delta_i} \omega_{\Delta}$$

Corollary 1.19 : $|\Omega| = [P : Q]$
$$= 1 + \#\{i : \langle \alpha_0, \varepsilon_i \rangle = 1$$

Corollary 1.20 : For any cell D,
$$\overline{D} \cap P^\perp = \{x\}.$$
In ptc. $\overline{D_0} \cap P^\perp = \{0\}$.

Pf : $P^\perp$ stable under $D'W$, $D'W$ acts transitively on cells, $\therefore$ ETS $\overline{D_0} \cap P^\perp = \{0\}$

$x \neq 0 \in \overline{D_0} \cap P^\perp$. $P^\perp \leq Q^\perp \implies$
$\exists\ i$ w/ $x = \varepsilon_i$ $(\alpha_0, \varepsilon_i) = 1$.
then $T(x) = T(\varepsilon_i) \in D'$ $(x \in P^\perp)$
$\therefore$ $T(\varepsilon_i) w_{\Delta_i} w_\Delta \in D'W \cap \Omega$
$$= \{I\}.$$
$$\Rightarrow\Leftarrow$$
$\square$

Lattice point := unique $\cap$ pt of $\overline{D} \cap P^\perp$

Observe : $\sigma, \tau \in D'W$
Then $\sigma(\overline{D_0} \cap P^\perp) = \sigma(\overline{D_0}) \cap \sigma(P^\perp)$
$= \sigma(0)$

If lattice pts assoc. to σD_0 τD_0 are same, then $\sigma(0) = \tau(0)$
$$\iff \sigma\tau^{-1}(0) = 0$$
$$\iff \sigma\tau^{-1} \in W$$
$$\iff \sigma W = \tau W.$$

§ 1.8

In this section we show that Ω acts on the affine Dynkin diagram.

We first consider the map $\Omega \longrightarrow \mathrm{Aut}(D'W)$

$$\rho \mapsto (\sigma \mapsto \rho\sigma\rho^{-1})$$

Since $\lambda(\sigma) = \lambda(\rho\sigma\rho^{-1})$, this automorphism induces a permutation of $\{w_0, .., w_\ell\}$

(Recall Megan showed that for $\sigma \in D'W$, $\lambda(\sigma) = \ell(\sigma)$ ("length") with $\ell(\sigma) = 1 \iff \sigma \in \{w_0, \dots, w_\ell\}$)

<u>Claim</u> : $\Omega \hookrightarrow S_{\ell+1}$

<u>Pf</u> : Let $\rho = T(\varepsilon_i) w_{\Delta_i} w_\Delta \in \Omega$. S'pose $\rho w_j \rho^{-1} = w_j \ \forall j$. Then

$$w_j T(\varepsilon_i) w_{\Delta_i} w_\Delta w_j^{-1} = T(\varepsilon_i) w_{\Delta_i} w_\Delta \ \forall j.$$

$$\Rightarrow w_j T(\varepsilon_i) w_j^{-1} \underbrace{w_j w_{\Delta_i} w_\Delta w_j^{-1}}_{= w_{\Delta_i} w_\Delta} = T(\varepsilon_i) w_{\Delta_i} w_\Delta$$

$w_j^{-1}(\Delta) = w_j^{-1}(\Delta \setminus \{\alpha_j\}) \cup \{-\alpha_j\}$

$w_\Delta w_j^{-1}(\Delta) = w_j^{-1}(-\Delta \setminus \{-\alpha_j\}) \cup \{\alpha_j\}$

$w_j w_\Delta w_j^{-1}(\Delta) = -\Delta$

$$\Rightarrow w_j T(\varepsilon_i) w_j^{-1} = T(\varepsilon_i) \quad \forall j$$

$$\Rightarrow w_j(\varepsilon_i) = \varepsilon_i \quad \forall j$$

$$\Rightarrow \langle \alpha_j, \varepsilon_i \rangle = 0 \quad \forall j \quad \Rightarrow \quad \varepsilon_i = 0 \quad \Rightarrow \Leftarrow$$

$\square$

<u>Proposition 1.21</u> : (i) Let $\rho = T(\varepsilon_i) \cdot w_{\Delta_i} \cdot w_\Delta \in \Omega$, $\langle \alpha_0, \varepsilon_i \rangle = 1$

Then $\rho w_0 \rho^{-1} = w_i$

(ii) Let $\varphi : DW \to W$ be natural hom.

Then φ is injective on Ω and the set $\{\alpha_1, \dots, \alpha_\ell, -\alpha_0\}$ is stable under the subgp $W_\Omega = \varphi(\Omega) \leq W$.

<u>Proof</u> : (i) First we show that $\rho w_0 \rho^{-1} \in W$, i.e. $\rho w_0 \rho^{-1}(0) = 0$.

(i.e $\rho^{-1}(0)$ lies on hyperplane $\langle \alpha_0, \cdot \rangle = 1$, and w_0 is reflection along this hyperplane, so $w_0 \rho^{-1}(0) = \rho^{-1}(0)$)

$\rho^{-1}(0) = w_\Delta w_{\Delta_i}(-\varepsilon_i)$ (inverse, $w_{\Delta_i}^2 = w_\Delta^2 = 1$)

for $j \neq i$ $\quad \underset{\uparrow \langle \alpha_j, \varepsilon_i \rangle = 0}{w_j(\varepsilon_i) = \varepsilon_i}$ so that $w_{\Delta_i}(\varepsilon_i) = \varepsilon_i$.

$$\therefore \quad \rho^{-1}(0) = w_\Delta(w_{\Delta_i}(-\varepsilon_i)) = w_\Delta(-\varepsilon_i) = -w_\Delta(\varepsilon_i)$$

Thus have
$$\langle \alpha_0, -w_\Delta(\varepsilon_i) \rangle = \langle w_\Delta(\alpha_0), -\varepsilon_i \rangle = \langle -\alpha_0, -\varepsilon_i \rangle = 1.$$

so $\rho w_0 \rho^{-1} \in W$ as reqd.

$\Rightarrow \rho w_0 \rho^{-1} \in \{w_1, \dots, w_\ell\} \subseteq W$ (not w_0)

Since $\Omega \cap D = \{1\}$, $\varphi : DW \to W$ is injective on Ω.

$\therefore$ enough to determine $\varphi(\rho w_0 \rho^{-1})$

$$\therefore \varphi(\rho w_0 \rho^{-1}) = T(\varepsilon_i) w_{\Delta_i} w_\Delta w_0 w_\Delta w_{\Delta_i} T(-\varepsilon_i)$$

$(w_0 = T(\check{\alpha}_0) \cdot w_{\alpha_0})$
$$= w_{\Delta_i} \underbrace{w_\Delta w_{\alpha_0} w_\Delta}_{= w_{w_\Delta(\alpha_0)} = w_{-\alpha_0} = w_{\alpha_0}} w_{\Delta_i}$$

$$= w_\beta \qquad \beta = w_{\Delta_i}(\alpha_0).$$

But $\beta \in \pm\Delta$ as $\rho w_0 \rho^{-1} \in \{w_1, \dots, w_\ell\}$

$(w_{-\alpha_1} = w_{\alpha_i})$

Before we saw $w_{\Delta_i}(\alpha_0) > 0$ so

$\beta \in \Delta$.

$\alpha_0 = \alpha_i + \sum_{j \neq i} m_j \alpha_j \qquad [\ (\alpha_0, \varepsilon_i) = 1)\ ]$

$\Rightarrow \ \beta = \alpha_i + \sum_{j \neq i} \mu_j \alpha_j \qquad [\ \Delta_i$ prod of w_j $j \neq i\ ]$

$\therefore \ \beta = \alpha_i$ and so $\rho w_0 \rho^{-1} = w_i$.

(ii) Consider non-triv $\rho = T(\varepsilon_i) w_{\Delta_i} w_\Delta \in \Omega$.

then $\varphi(\rho)(-\alpha_0^\vee) = w_{\Delta_i} w_\Delta (-\alpha_0^\vee) = \alpha_i^\vee$

$\rho^{-1} \in \Omega$ so write $\rho^{-1} = T(\varepsilon_j) w_{\Delta_j} w_\Delta$

$\rightsquigarrow w_{\Delta_j} w_\Delta = (w_{\Delta_i} w_\Delta)^{-1} = w_\Delta w_{\Delta_i}$

$\therefore \ \varphi(\rho)(\alpha_j) = w_{\Delta_i} w_\Delta (\alpha_j)$

$= w_\Delta w_{\Delta_j}(\alpha_j) = -\alpha_0$

(as $w_{\Delta_j} w_\Delta(-\alpha_0) = \alpha_j$)

also for $k \in \Delta \setminus \{\alpha_j\}$

$\varphi(\rho)(\alpha_k) = w_\Delta w_{\Delta_j}(\alpha_k)$

$\in w_\Delta(-\Delta_j) \subset \Delta$

$\therefore \ \varphi(\rho)$ keeps $\{\alpha_1, \dots, \alpha_\ell, -\alpha_0\}$ stable

D

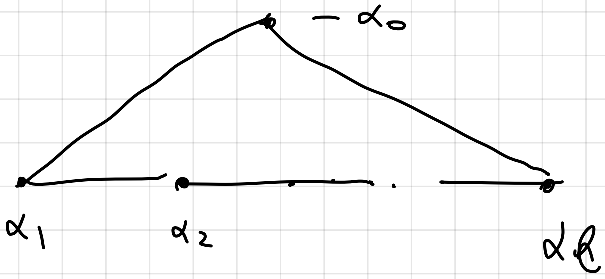
<u>Corollary</u>: $\langle \alpha_0, \varepsilon_i \rangle = 1 \implies w_{D_i}(\alpha_0) = \alpha_i.$

<u>Rmk</u>: The permutation $w_i \longmapsto \rho w_i \rho^{-1}$ $0 \le i \le \ell$ of the set $\{w_0, w_1, \dots, w_\ell\}$ induced by $\rho \in \Omega$ coincides with the permutation of the Dynkin diagram of $\{-\alpha_0, \dots, \alpha_\ell\}$ induced by $\varphi(\rho) \in W_\Omega \subset W$

pairing is W-invariant

Moreover $\varphi(\rho)$ preserves the angle between $-\alpha_0, \alpha_1, \dots, \alpha_\ell$ .. so $\varphi(\rho)$ is an automorphism of the Dynkin diagram of $\{-\alpha_0, \dots, \alpha_\ell\}$

<u>Ex</u>. A_n



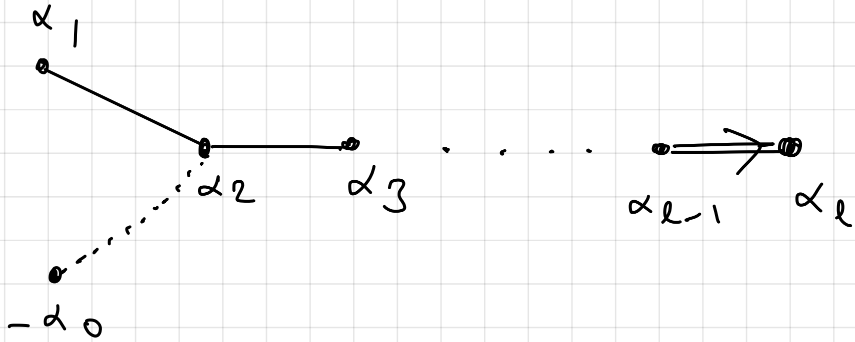
$\alpha_0 = \alpha_1 + \cdots + \alpha_\ell$

$\Omega \simeq \mathbb{Z}_{\ell+1}$ [cyclic] $\le W \simeq S_{\ell+1}.$

$\rho = T(\varepsilon_1) w_{\Delta_1} w_D$ generates Ω

$\rho w_0 \rho^{-1} = w_1$ $\rho w_1 \rho^{-1} = w_2$ $\dots$ $\rho w_\ell \rho^{-1} = w_0$

$\underline{B_n}$



$$\alpha_0 = \alpha_1 + 2(\alpha_2 + \cdots + \alpha_l)$$

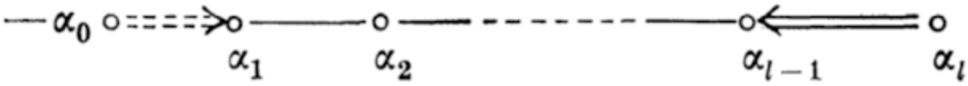
$$\Omega \simeq \mathbb{Z}_2 \qquad \Omega = \{1, \rho\}$$

$$\rho = T(\varepsilon_1)\, w_{\Delta_1, \Delta}$$

$$\rho w_0 \rho^{-1} = w_1 \qquad \rho w_1 \rho^{-1} = w_0$$

$$\rho w_i \rho^{-1} = w_i$$

$\underline{C_l}$



$$\alpha_0 = 2(\alpha_1 + \ldots + \alpha_{l-1}) + \alpha_l$$

$$\Omega \cong \mathbf{Z}_2, \quad \Omega = \{1, \rho\}, \quad \rho = T(\varepsilon_l)\, w_{\Pi_l} w_{\Pi}.$$

$$\rho w_0 \rho^{-1} = w_l, \quad \rho w_1 \rho^{-1} = w_{l-1}, \ \ldots, \ \rho w_l \rho^{-1} = w_0.$$

etc