

ℓ -parity results over global function fields

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Global Function Fields

- Let $K = \mathbb{F}_q(C)$ be the function field of a smooth, projective, geometrically irreducible curve C/\mathbb{F}_q , with $q = p^k$.
- A closed point of C is a $\text{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ -orbit of a point $P \in C(\overline{\mathbb{F}}_q)$, and places of $K \leftrightarrow$ closed points of C .
- For a closed point x ,

\mathcal{O}_x	local ring of functions that are regular at x ,
$\mathfrak{m}_x \subset \mathcal{O}_x$	maximal ideal of functions vanishing at x ,
$\kappa_x = \mathcal{O}_x/\mathfrak{m}_x$	residue field with $\deg(x) := [\kappa_x : \mathbb{F}_q]$,
K_x	completion at x .

Example

Consider $C = \mathbb{P}_{\mathbb{F}_q}^1$ with function field $K = \mathbb{F}_q(t)$.

- The closed points of C correspond to monic irreducible polynomials in $\mathbb{F}_q[t]$, together with the point at infinity.
- For a closed point x corresponding to an irreducible polynomial $f(t)$:

$$\mathcal{O}_x = \left\{ \frac{g}{h} \in \mathbb{F}_q(t) : f \nmid h \right\}, \quad \mathfrak{m}_x = (f), \quad \kappa_x = \mathbb{F}_q[t]/(f) \simeq \mathbb{F}_{q^{\deg(f)}}, \quad K_x = \kappa_x((f))$$

- The point at infinity corresponds to the valuation $v_\infty(f/g) = \deg(g) - \deg(f)$.

- An abelian variety A/K is a complete connected algebraic group.
- In other words, it's an elliptic curve, the Jacobian of a smooth projective curve, or something else.
- Mordell–Weil theorem: the group of K -rational points $A(K)$ is a finitely generated abelian group:

$$A(K) \simeq A(K)_{\text{tors}} \oplus \mathbb{Z}^r, \quad r = \text{rk}(A/K).$$

- For $\ell \neq p$, the ℓ -adic Tate module:

$$T_\ell(A) = \varprojlim_n A[\ell^n](K^{\text{sep}}), \quad V_\ell(A) = T_\ell(A) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell,$$

carries a continuous $G_K = \text{Gal}(K^{\text{sep}}/K)$ -action.

L -functions and BSD

Given an abelian variety A/K , its Hasse–Weil L -function is given by an Euler product ($\ell \neq p$):

$$L(A/K, s) = \prod_x P_x(q^{(1-s) \cdot \deg(x)})^{-1}, \quad P_x(T) = \det(1 - \varphi_x T \mid V_\ell(A)^{I_x}).$$

where

- x ranges over all closed points of C ,
- $I_x \leq G_K$ is the inertia subgroup at x ,
- φ_x is the geometric Frobenius at x , acting on $\bar{\kappa}_x$ by $y \mapsto y^{1/\#\kappa_x}$.

Converges absolutely for $\operatorname{Re}(s) > \frac{3}{2}$ (Weil bounds).

Birch–Swinnerton-Dyer Conjecture

$$\operatorname{rk} A/K = \operatorname{ord}_{s=1} L(A/K, s).$$

BSD over function fields fact file

- Unbounded rank: There exist explicit families of elliptic curves over function fields with arbitrarily large rank.
- Tate: $\text{rk } A/K \leq \text{ord}_{s=1} L(A/K, s)$.
- BSD for Jacobians \implies BSD for all abelian varieties.
- Kato–Trihan (2003): BSD is equivalent to showing the finiteness of any ℓ -primary part of the Tate-Shafarevich group $\text{III}(A/K)$.
- Kato–Trihan (2003): Weak BSD implies refined BSD.
- BSD for Jacobians: look at surface over \mathbb{F}_q corresponding to curve. Néron-Severi groups, III is a Brauer group, Artin–Tate conjecture for surfaces ...

L-function of an elliptic curve

Example

Consider $E: y^2 = x^3 + t^2x + t$ over $K = \mathbb{F}_3(t) = \mathbb{F}_3(\mathbb{P}^1)$. Then E has bad reduction at t and at ∞ , and good reduction elsewhere. The L -function of E can be computed using the formula^a

$$L(E/K, T) = \frac{Z(C, T)Z(C, qT)}{Z(\mathcal{E}, T)} \prod_{\text{bad } v} \frac{(1 - T)^{a_v+1}(1 + T)^{b_v}}{(1 - q_v T^{\deg(v)})^{f_v-1}(1 + q_v T^{\deg(v)})^{g_v}},$$

where \mathcal{E}/\mathbb{F}_3 is the Kodaira–Néron elliptic surface associated to E . One computes

$$L(E/K, T) = 1 - 18T^2 + 81T^4, \quad L(E/K, 3^{-s}) = 4(\log 3)^2 \cdot (s-1)^2 - 8(\log 3)^3 \cdot (s-1)^3 + \dots$$

Thus $\text{ord}_{s=1} L(E/K, s) = 2$.

^aProposition 6.1, *Elliptic curves over function fields*, Douglas Ulmer.

L-functions are rational functions

Theorem (Grothendieck)

$$L(A/K, s) = \frac{L_1(s)}{L_0(s) \cdot L_2(s)}, \quad L_i(s) = \det(1 - q^{1-s} \varphi_\ell, \mathcal{H}_{\mathbb{Q}_\ell}^i) \in \mathbb{Q}[q^{-s}],$$

where $\mathcal{H}_{\mathbb{Q}_\ell}^i$ are étale cohomology groups with an action of φ_ℓ induced by geometric Frobenius.

Idea: Representations \leftrightarrow sheaves correspondence and Grothendieck-Lefschetz trace formula.

- It follows that $L(A/K, s)$ is a rational function in q^{-s} and admits a meromorphic continuation to \mathbb{C} .
- Duality results $\rightsquigarrow L(A/K, s)$ satisfies a functional equation centered at $s = 1$.
- Deligne: Eigenvalues of φ_ℓ acting on $\mathcal{H}_{\mathbb{Q}_\ell}^i$ have absolute value $q^{(i-1)/2}$.
- Thus

$$\begin{aligned} \text{ord}_{s=1} L(A/K, s) &= \text{multiplicity of } 1 \text{ as a root of characteristic polynomial of } \varphi_\ell \\ &= \text{dimension of part of } \mathcal{H}_{\mathbb{Q}_\ell}^1 \text{ on which } \varphi_\ell \text{ acts unipotently.} \end{aligned}$$

The Parity Conjecture

Functional equation

$$L(A/K, 2-s) = w(A/K) \cdot q^{\alpha(s)} \cdot L(A/K, s).$$

The global root number $w(A/K) \in \{\pm 1\}$ is the sign appearing in the functional equation, which can be computed as a product of local root numbers.

When K is a global function field,

$$w(A/K) = (-1)^{\text{ord}_{s=1} L(A/K, s)}.$$

Parity conjecture

$$w(A/K) = (-1)^{\text{rk } A/K}.$$

In other words,

$$\text{ord}_{s=1} L(A/K, s) \equiv \text{rk } A/K \pmod{2}.$$

ℓ -parity conjecture

Define the ℓ^∞ -Selmer group $\mathrm{Sel}_{\ell^\infty}(A/K) = \varinjlim_n \mathrm{Sel}_{\ell^n}(A/K)$ where

$$\mathrm{Sel}_{\ell^n}(A/K) = \ker(H^1(G_K, A[\ell^n]) \rightarrow \prod_v H^1(G_{K_v}, A(K_v^{\mathrm{sep}}))).$$

This fits into the following exact sequence of cofinitely generated \mathbb{Z}_ℓ -modules.

$$0 \rightarrow A(K) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell \rightarrow \mathrm{Sel}_{\ell^\infty}(A/K) \rightarrow \mathrm{III}(A/K)[\ell^\infty] \rightarrow 0.$$

Set $\mathcal{X}_\ell(A/K) = \mathrm{Hom}(\mathrm{Sel}_{\ell^\infty}(A/K), \mathbb{Q}_\ell/\mathbb{Z}_\ell) \otimes \mathbb{Q}_\ell$. The dimension of $\mathcal{X}_\ell(A/K)$ is the Mordell–Weil rank of A/K plus the number of copies of $\mathbb{Q}_\ell/\mathbb{Z}_\ell$ in $\mathrm{III}(A/K)$.

ℓ -parity conjecture

$$w(A/K) = (-1)^{\dim_{\mathbb{Q}_\ell} \mathcal{X}_\ell(A/K)}.$$

If $|\mathrm{III}(A/K)[\ell^\infty]| < \infty$ then ℓ -parity conjecture \implies parity conjecture.

Semi-simplicity. What, like it's hard?

Let $r_\ell = \dim_{\mathbb{Q}_\ell} \mathcal{X}_\ell(A/K)$. By work of Kato and Trihan, it turns out that

$$r_\ell = \dim_{\mathbb{Q}_\ell} \ker(1 - \varphi_\ell \mid \mathcal{H}_{\mathbb{Q}_\ell}^1).$$

Set

$$\mathcal{I}_{2,\ell} = \ker(1 - \varphi_\ell \mid \mathcal{H}_{\mathbb{Q}_\ell}^1), \quad \mathcal{I}_{3,\ell} = \text{part of } \mathcal{H}_{\mathbb{Q}_\ell}^1 \text{ where } \varphi_\ell \text{ acts unipotently.}$$

Recall that $\text{ord}_{s=1} L(A/K, s) = \dim_{\mathbb{Q}_\ell} \mathcal{I}_{3,\ell}$. One has $\mathcal{I}_{2,\ell} \subset \mathcal{I}_{3,\ell}$ with equality iff. φ_ℓ acts semi-simply on $\mathcal{H}_{\mathbb{Q}_\ell}^1$.

Theorem (Trihan–Yasuda (2014))

The ℓ -parity conjecture for abelian varieties is true over global function fields for any prime ℓ . In other words,

$$\dim_{\mathbb{Q}_\ell} \mathcal{I}_{2,\ell} \equiv \dim_{\mathbb{Q}_\ell} \mathcal{I}_{3,\ell} \pmod{2}.$$

Trihan and Yasuda construct a perfect pairing

$$(\cdot, \cdot)_\ell: \mathcal{I}_{3,\ell} \times \mathcal{I}_{3,\ell} \rightarrow \mathbb{Q}_\ell.$$

that is symmetric and compatible with Frobenius action φ_ℓ . This comes from the Weil-pairing on $V_\ell(A)$.

Thus one can view φ_ℓ as a unipotent element of the orthogonal group $O((\cdot, \cdot)_\ell)$. Linear algebra says that such a unipotent element satisfies

$$\det(-\varphi_\ell) = (-1)^{\dim \ker(1-\varphi_\ell)}.$$

Trihan-Yasuda's more general parity result

The exact same argument to deduce the ℓ -parity conjecture for abelian varieties yields the following result.

Theorem

Let F_ℓ be a smooth $\overline{\mathbb{Q}_\ell}$ -sheaf on an open subset $U \subset C$. Assume that F_ℓ is pure of weight -1 and equipped with a skew-symmetric non-degenerate pairing $F_\ell \times F_\ell \rightarrow \overline{\mathbb{Q}_\ell}(1)$. Then

$$r_{an}(F_\ell) \equiv r(F_\ell) \pmod{2}.$$

Here $r_{an}(F_\ell)$ is the order of vanishing at $s = 1$ of the L -function $L(U, F_\ell, s)$, and $r(F_\ell) = \ker(1 - \varphi_\ell \mid \mathcal{H}_{\overline{\mathbb{Q}_\ell}}^1(F_\ell))$.

In other words, if you have

- A continuous ℓ -adic representation $\tau: G_K \rightarrow \mathrm{GL}_n(\overline{\mathbb{Q}_\ell})$ that is unramified outside a finite number of places.
- For all places v where τ is unramified, the eigenvalues of $\tau(\varphi_v)$ have absolute value $\kappa_v^{-1/2}$.
- τ has a G_K -equivariant skew-symmetric pairing $\tau \times \tau \rightarrow \overline{\mathbb{Q}_\ell}(1)$.

Then you get a parity-like result for τ .

Application: ℓ -parity for twists

Consider a Galois extension F/K with Galois group G and an abelian variety A/K . Then the L -function $L(A/F, s)$ factors into a product of twisted L -functions

$$L(A/F, s) = \prod_{\rho \in \text{Irr}_{\mathbb{C}}(G)} L(A, \rho, s)^{\dim \rho}.$$

The group $A(F) \otimes_{\mathbb{Z}} \mathbb{C}$ is a G -representation, and the analogue of BSD for twists conjectures that

$$\langle \rho, A(F) \otimes_{\mathbb{Z}} \mathbb{C} \rangle = \text{ord}_{s=1} L(A, \rho, s),$$

for a representation ρ of G .

Similarly, $\mathcal{X}_{\ell}(A/F)$ inherits a G -action, and the ℓ -parity conjecture for twists states that

$$w(A, \rho) = (-1)^{\text{ord}_{s=1} L(A, \rho, s)} = (-1)^{\langle \rho, \mathcal{X}_{\ell}(A/F) \rangle}.$$

Corollary

The ℓ -parity conjecture for twists of abelian varieties by orthogonal Artin representations holds for all primes $\ell \neq p$.

Proof sketch

Let ρ be an orthogonal Artin representation factoring through a finite Galois extension F/K , with $G = \text{Gal}(F/K)$. View this as a continuous ℓ -adic representation.

Step 1: Find a suitable representation

- Set $W = \text{Res}_K^F A$, the Weil restriction of A/F . This is an abelian variety of dimension $[L: K] \cdot \dim A$. The group G acts on W by K -automorphisms, and we can consider $V_\ell(W)$ as a $G \times G_K$ -representation.
- Let $\sigma = \text{Hom}_G(\rho, V_\ell(W))$, where G_K acts trivially on ρ .
- Then σ is unramified outside a finite set of places, and when v is unramified, $\sigma(\varphi_v)$ has eigenvalues of absolute value $\kappa_v^{-1/2}$.
- As ρ is orthogonal, it is equipped with a symmetric G_K -equivariant pairing $\rho \times \rho \rightarrow \overline{\mathbb{Q}}_\ell$ and so σ has a G_K -equivariant skew-symmetric pairing $\sigma \times \sigma \rightarrow \overline{\mathbb{Q}}_\ell(1)$.

Trihan–Yasuda:

$$r_{\text{an}}(F_{\sigma, \ell}) \equiv r(F_{\sigma, \ell}) \pmod{2},$$

where $F_{\sigma, \ell}$ is the sheaf corresponding to σ .

Step 2: Translate.

- Show $r_{\text{an}}(F_{\sigma,\ell}) = \text{ord}_{s=1} L(A, \rho, s)$.
- Show $r(F_{\sigma,\ell}) = \langle \rho, \mathcal{X}_{\ell}(W/K) \rangle$.
- Back to A/F : $\mathcal{X}_{\ell}(W/K) \simeq \mathcal{X}_{\ell}(A/F)$ as G -representations.

So

$$\text{ord}_{s=1} L(A, \rho, s) \equiv \langle \rho, \mathcal{X}_{\ell}(A/F) \rangle \pmod{2},$$

as required.

Why did I want this?

A better question: Why did Vladimir Dokchitser, Holly Green and Adam Morgan want this?

Vague idea: 'Local to global' methods. Know

$$w(A/K, \rho) = (-1)^{\rho, \chi_\ell(A/F)}.$$

The LHS is $\prod_v w(A/K_v, \text{Res}_{D_v} \rho)$.

Suppose we could write the RHS as $\prod_v S_{K_v}$ where $S_{K_v} \in \{\pm 1\}$ is a function of A/K_v .

Local formula:

$$w(A/\mathcal{K}, \tau) \stackrel{?}{=} S_{\mathcal{K}} \cdot h_{\mathcal{K}},$$

where \mathcal{K} is a local field of positive characteristic and τ is an Artin representation of \mathcal{K} . The term $h_{\mathcal{K}} \in \{\pm 1\}$ should be an error term such that when K is a global field, $\prod_v h_{K_v} = 1$.

Using the global parity result, we can prove the local formula for A/\mathcal{K} with bad reduction if we can approximate it by a global abelian variety \tilde{A}/K such that $K_w = \mathcal{K}$ for some place w and $S_{\mathcal{K}}(A) = S_{K_w}(\tilde{A})$, and \tilde{A}/K_v satisfies the local formula at all $v \neq w$.

Thank you for your attention!