Simple wild parameters and simple supercuspidal representations

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Introduction

Let p be a prime. The local reciprocity map

$$\rho_{\mathbb{Q}_p} \colon \mathbb{Q}_p^\times \to \mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ab}}/\mathbb{Q}_p)$$

of local class field theory establishes a bijection between finite abelian Galois extensions $F/\mathbb{Q}_p$ and finite index subgroups of $\mathbb{Q}_p^\times$, given by mapping F to its norm subgroup $N_{F/\mathbb{Q}_p}(F^\times) \subset \mathbb{Q}_p^\times$. Thus there is a correspondence between characters of $\mathrm{Gal}(F/\mathbb{Q}_p)$ and characters of $\mathbb{Q}_p^\times$ that are trivial on $N_{F/\mathbb{Q}_p}(F^\times)$. The image of $\mathbb{Q}_p^\times$ under $\rho_{\mathbb{Q}_p}$ is isomorphic to the abelianization $W_{\mathbb{Q}_p}^{\mathrm{ab}}$ of the Weil group $W_{\mathbb{Q}_p}$, which is a dense subgroup of the absolute Galois group $G_{\mathbb{Q}_p}$. Therefore we also have the following correspondence

$$\left\{ \text{ characters } \chi \colon \mathbb{Q}_p^\times \to \mathbb{C}^\times \right\} \leftrightarrow \left\{ \text{ characters } \psi \colon W_{\mathbb{Q}_p} \to \mathbb{C}^\times \right\}.$$

The local Langlands conjecture (LLC) generalizes this from $\mathbb{Q}_p^\times = \mathrm{GL}_1(\mathbb{Q}_p)$ to reductive groups over local fields. Let G be a split connected reductive group over a local field k. The LLC predicts that there is a finite-to-one map

$$\left\{ \begin{array}{c} \text{irreducible discrete series} \\ \text{representations of } G(k) \end{array} \right\} \to \left\{ \begin{array}{c} \text{Langlands parameters} \\ \varphi\colon W_k \times \mathrm{SL}_2(\mathbb{C}) \to \hat{G} \end{array} \right\}.$$

The left hand side consists of certain complex representations of $G(k)$. On the right hand side, we consider homomorphisms from $W_k \times \mathrm{SL}_2(\mathbb{C})$ instead of from W_k as in the $\mathrm{GL}_1(k)$ case, and we replace $\mathbb{C}^\times$ by $\hat{G}$, which is a complex Lie group with dual root datum to that of G.

The LLC has been proven for general linear groups and in many cases for classical groups. Vaguely, to establish a correspondence, one finds all the irreducible discrete series representations of $G(k)$ and then proves that these correspond to Langlands parameters in a unique way. Unfortunately, this is not always explicit.

In this report we focus on an explicit instance of the local Langlands correspondence, which was set up by Gross and Reeder in [GR10] and later shown to be a correspondence by Kaletha in [Kal13]. This correspondence is between simple supercuspidal representations and simple wild parameters. Simple supercuspidal representations are a class of representations characterized by their construction through compact induction from characters of open compact subgroups, while simple wild parameters are specific types of Langlands parameters that exhibit wild ramification.

We adopt a hands-on approach to showcasing the theory by working it out explicitly for $\mathrm{SL}_2(\mathbb{Q}_p)$. The main calculations of this report are devoted to computing the simple wild parameters and simple supercuspidal representations for $\mathrm{SL}_2(\mathbb{Q}_p)$, and then trying to match them up.

We hope that this report provides an illuminating example of the LLC, where one can really see and understand the objects that match up. In addition, it provides an example of the correspondence that can be understood by the reader who is largely unfamiliar with the local Langlands correspondence.

Layout of report

This report introduces the theory of simple supercuspidal representations and simple wild parameters, as well as introducing necessary background material in order to compute these objects for our case of interest $\mathrm{SL}_2(\mathbb{Q}_p)$.

In the first section, we describe the ramification groups of the Galois group of an extension of local fields and discuss their properties. We define a numerical invariant known as the Swan conductor of a representation of a local Galois group. We review local class field theory and briefly discuss Weil-Deligne representations.

In the second section, we define Langlands parameters. We discuss different types of Langlands parameters and focus on simple wild parameters, whose definition involves the Swan conductor. We then compute all simple wild parameters for $\mathrm{SL}_2(\mathbb{Q}_p)$ for varying primes p.

In the following section, we introduce some Bruhat-Tits theory which allows us to consider special subgroups of p-adic groups known as parahoric subgroups. These become relevant in the construction of simple supercuspidal representations, which we discuss in §4. We detail the construction, and then compute these representations for SL_2 over unramified extensions of $\mathbb{Q}_p$.

In the last section, we discuss the local Langlands conjecture, focusing on our particular case. We describe the so-called L-packets of our simple supercuspidal representations for $\mathrm{SL}_2(\mathbb{Q}_p)$, and consider the correspondence over the quadratic unramified extension $\mathbb{Q}_{p^2}$ of $\mathbb{Q}_p$ to understand how to match our simple supercuspidal representations with simple wild parameters.

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1 Local Galois theory

1.1 Local fields and class field theory

In this section we recall some concepts about ramification theory of local fields and local class field theory that will be useful in later calculations. We also define the swan conductor. A good reference is [Iwa86].

Throughout this section, we let k be a non-archimedean local field of characteristic zero, with residue characteristic p. Let k have normalized discrete valuation v_k, ring of ideals $\mathcal{O}_k$, maximal ideal $\mathfrak{p}_k$, and residue field $\mathfrak{f}_k = \mathcal{O}_k/\mathfrak{p}_k$ of order q. Choose a uniformizer $\varpi \in \mathcal{O}_k$ such that $v_k(\varpi) = 1$.

1.1.1 Ramification groups and Swan conductors

See [Ser79, Chapter IV] for more details on ramification groups.

Let F/k be a finite Galois extension, with $H = \mathrm{Gal}(F/k)$. Let v_F, $\mathcal{O}_F$, $\mathfrak{p}_F$, $\mathfrak{f}_F$ be defined as above for F. In particular, $v_F(x) = e_{F/k} \cdot v_k(x)$ for $x \in k$, where $e_{F/k}$ is the ramification degree. Let ϖ_F be a uniformizer of F. The lower ramification groups provide a filtration of H depending on the ramification of the extension.

Definition 1.1 (Lower ramification groups)**.** Let $s \in \mathbb{R}_{\geq -1}$. Define subgroups $G_s(F/k) \subset H$ by

$$G_s(F/k) = \{\sigma \in G \mid \forall x \in \mathcal{O}_F,\ v_F(\sigma(x) - x) \geq s + 1\}.$$

When it is clear by context, we will commonly denote these groups by G_s.

Observe that, since the valuation on F is discrete and normalized, $G_s = G_{\lceil s \rceil}$, so the values only change at integers. One has that $G_{-1} = \mathrm{Gal}(F/k)$.

Definition 1.2 (Inertia and wild inertia subgroups)**.** The subgroup

$$I_{F/k} := G_0 = \{\sigma \in \mathrm{Gal}(F/k) \colon \forall x \in \mathcal{O}_F,\ \sigma(x) \equiv x \pmod{\mathfrak{p}_F}\}$$

is the *inertia subgroup.* This is the kernel of the map $H \to \mathrm{Gal}(\mathfrak{f}_F \,/\, \mathfrak{f}_k)$ surjecting to the Galois group of the residue fields. The subgroup G_1 is the *wild inertia subgroup.*

One also has that $I_{F/k} = \mathrm{Gal}(F/k')$ where k' is the maximal unramified subextension of F/k. Thus if $I_{F/k} = \{1\}$ then F/k is *unramified.* We say that F/k is *tamely ramified* if $p \nmid e_{F/k}$, which is equivalent to having $G_1 = \{1\}$. Otherwise, it is *wildly ramified.* One has $G_1 = \mathrm{Gal}(F/k'')$ where k''/k is the maximal tamely ramified subextension of F/k.

For $s \geq 1$, G_s is the kernel of the action of G_0 on $\mathcal{O}_F/\mathfrak{p}_F^{s+1}$. Therefore each G_s is normal in G_0, and we obtain a finite filtration of G by normal subgroups:

$$G = G_{-1} \geq G_0 \geq G_1 \geq \cdots \geq G_c \gneq G_{c+1} = 1,$$

with $c \in \mathbb{Z}$. The following proposition details the order of the quotient groups in this filtration.

Proposition 1.3 ([Ser79, Ch IV §2, Corr 1, 2])**.** *Let F/k be a finite Galois extension, with ramification index $e_{F/k}$ and residue degree $f_{F/k}$.*

1. *The quotient G_{-1}/G_0 is isomorphic to* $\mathrm{Gal}(\mathfrak{f}_F \,/\, \mathfrak{f}_k)$*. Thus it is cyclic of order $f_{F/k}$.*
2. *The quotient G_0/G_1 is cyclic of order prime to p.*
3. *If $G_j \neq 1$, then G_j/G_{j+1} is an elementary p-group.*

The lower ramification groups behave well with respect to subgroups of G.

Proposition 1.4 ([Ser79, Ch IV §1 Proposition 2])**.** *Let $L/F/k$ be finite extensions of non-archimedean local fields, and let L/k be Galois. Then for $s \in \mathbb{R}_{\geq -1}$ one has*

$$G_s(L/F) = G_s(L/k) \cap \mathrm{Gal}(L/F).$$

On the other hand, they do not behave well with respect to taking quotients. One remedies this by defining the upper ramification groups, which are a renumbering of the lower ones. They are also relevant to us as they appear in some statements of local class field theory.

Let F/k be as above. To define the upper ramification groups, first define $\phi_{F/k} \colon \mathbb{R}_{\geq -1} \to \mathbb{R}$ by

$$s \mapsto \int_0^s \frac{1}{[G_0 \colon G_t]} dt.$$

By convention, for $t = -1$ we let $[G_0 \colon G_t] = [G_{-1} \colon G_0]^{-1}$, and for $-1 < t \leq 0$, $[G_0 \colon G_t] = 1$. More explicitly, if $m \leq s < m + 1$, then

$$\phi_{F/k}(s) = \begin{cases} s & m = -1, \\ \frac{1}{|G_0|} \left(|G_1| + \cdots + |G_m| + (s-m)|G_{m+1}|\right) & m \geq 0. \end{cases}$$

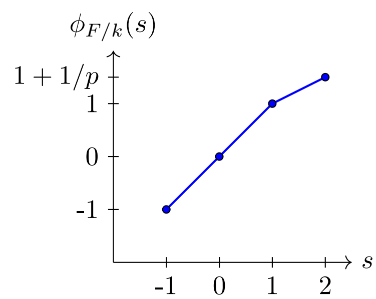
Note $\phi_{F/k}$ is continuous and strictly increasing, therefore we can define an inverse $\psi_{F/k}$, which is also strictly increasing. We use this inverse function to define the upper ramification groups.

Definition 1.5 (Upper ramification groups)**.** Let F/k be as above. For $s \in \mathbb{R}_{\geq -1}$, define

$$G^s(F/k) = G_{\psi_{F/k}(s)}(F/k).$$

Again, we also denote these by G^s when the extension referred to is clear.

Example 1.6. Let F/k be a degree p extension with $\mathrm{Gal}(F/k) = C_p$. Suppose that the lower ramification groups are $G_0 = C_p$, $G_1 = C_p$, $G_2 = C_1$. Then the graph of $\phi_{F/k}$ is



from which one can read off $\psi_{F/k}$ and compute that $G^s = C_p$ for $-1 \leq s \leq 1$ and $G^s = C_1$ for $s > 1$.

As promised, the upper ramification groups behave nicely with respect to quotients.

Proposition 1.7 ([Ser79, Ch IV §3 Lemma 5])**.** *Let $L/F/k$ be an extension of non-Archimedean local fields such that L/k and F/k are Galois. Let $J = \mathrm{Gal}(L/F)$. Then*

$$G^s(F/k) = G^s(L/k)J/J.$$

We say that there is an *upper ramification break* at v if $G^v \neq G^{v+\epsilon}$ for all $\epsilon > 0$. The following theorem says that in the case of abelian extensions, these breaks can only occur at integers.

Theorem 1.8 (Hasse–Arf, [Ser79, Ch V §7 Theorem 1])**.** *If $\mathrm{Gal}(F/k)$ is abelian and v is a ramification break, then v is an integer. In other words, if $G_i \neq G_{i+1}$, then $\phi_{F/k}(i)$ is an integer.*

The lower ramification groups appear in the definition of the Swan conductor of a representation. This will be an important numerical invariant of a representation in later sections. Let F/k be a finite Galois extension of k.

Definition 1.9 (Swan conductor)**.** The *Swan conductor* of a finite-dimensional representation $\rho\colon H \to \mathrm{GL}(V)$ of $H = \mathrm{Gal}(F/k)$ is

$$\mathrm{sw}(\rho) = \sum_{i=1}^{c} \frac{1}{[G_0\colon G_i]} \cdot \left(\dim V - \dim V^{G_i}\right),$$

where G_i denote the lower ramification groups of H, and $c \in \mathbb{Z}$ is such that $G_{c+1} = \{1\}$.

If G_1 acts trivially on V, then $\mathrm{sw}(\rho) = 0$. In this sense the Swan conductor measures the action of wild inertia. It turns out that the Swan conductor is an integer.

Theorem 1.10 (Artin, [Ser79, Ch VI, §2 Theorem 1']). *For any finite-dimensional representation $\rho\colon H \to \mathrm{GL}(V)$, $\mathrm{sw}(\rho)$ is an integer.*

1.1.2 Local class field theory

The following results of local class field theory are expanded upon in [Iwa86, Chapter 7]. Firstly, we recall the structure of the multiplicative group $k^\times$.

Proposition 1.11 ([Iwa86, §2.2]). *The multiplicative group $k^\times$ has a canonical decomposition*

$$k^\times \simeq \langle \varpi \rangle \times \mu_{q-1} \times (1 + \mathfrak{p}_k),$$

where $q = |\,\mathfrak{f}_k\,|$ is the order of the residue field, and μ_{q-1} denotes the $(q-1)$-th roots of unity in k. This decomposition depends on ϖ.

An extension F/k is called *abelian* if it is Galois and $\mathrm{Gal}(F/k)$ is an abelian group. As the compositum of abelian extensions is abelian, there exists a maximal abelian extension of k which we denote by k^{ab}. Recall that for a Galois extension F/k, $\mathrm{Frob}_{F/k} \in \mathrm{Gal}(F/k)$ is any element that maps to the Frobenius element in $\mathrm{Gal}(\mathfrak{f}_F\,/\,\mathfrak{f}_k)$ given by $x \mapsto x^{|\,\mathfrak{f}_k\,|}$ for $x \in \mathfrak{f}_F$. In particular this element is unique when k'/k is unramified and abelian. The main statement of local class field theory is the following:

Theorem 1.12 (Local Arin reciprocity, [Iwa86, Theorem 7.1]). *There exists a unique homomorphism $\rho_k\colon k^\times \to \mathrm{Gal}(k^{\mathrm{ab}}/k)$ satisfying the following properties:*

1. *For any uniformizer $\varpi \in k$, $\rho_k(\varpi)|_{k'} = \mathrm{Frob}_{k'/k}$ for any unramified extension k'/k.*
2. *If F/k is finite abelian, then ρ_k induces an isomorphism*

$$\rho_{F/k}\colon k^\times/N_{F/k}(F^\times) \xrightarrow{\simeq} \mathrm{Gal}(F/k).$$

We call a subgroup $N \subset k^\times$ a *norm group* if $N = N_{F/k}(F^\times)$ for some finite abelian extension F/k. The next proposition states that abelian extensions correspond to finite index subgroups of $k^\times$.

Proposition 1.13. *Let k be a local field of characteristic zero. Then every finite index subgroup of $k^\times$ is a norm group and there is an inclusion-reversing bijection*

$$\{ \textit{ finite abelian extensions of } k^\times \ \} \leftrightarrow \{ \textit{ finite index subgroups of } k^\times \ \}$$

given by sending F/k to $N_{F/k}(F^\times) \subset k^\times$.

One can also describe the images of the higher unit groups $(1+\mathfrak{p}_k^i)$ via the upper ramification groups.

Theorem 1.14 ([Iwa86, Theorem 7.12])**.** *Let F/k be a finite abelian extension and $\rho_{F/k}\colon k^\times/N_{F/k}(F^\times) \to \mathrm{Gal}(F/k)$ the isomorphism induced by the Artin map ρ_k. Then one has*

$$G^s(F/k) = \rho_{F/k}(1+\mathfrak{p}_k^i), \quad \text{for } i-1 \leq s \leq i,\ i \in \mathbb{Z}_{\geq 0}.$$

We will use the following lemma in later sections to show when a tower of Galois extensions is Galois.

Lemma 1.15. *Let L/k be Galois, and F/L an abelian Galois extension. If $\sigma\left(N_{F/L}(F^\times)\right) = N_{F/L}(F^\times)$ for every $\sigma \in \mathrm{Gal}(L/k)$, then F/k is Galois.*

Proof. To show that F/k is Galois, we need to show that there are $[F\colon k]$ k-automorphisms of F. There are $[F\colon L]$ L-automorphisms of F since F/L is Galois. If every k-automorphism of L extends to a k-automorphism of F, then we have enough k-automorphisms. Write $F = L[x]/(P(x))$ where $P \in L[x]$ is irreducible. Let $F' = L[x]/(\sigma(P(x)))$, where $\sigma(P(x)) \in L[x]$ is the polynomial obtained by applying σ to the coefficients of $P(x)$. Let $\sigma\colon F \to F'$ be the natural map.

We show that $\sigma(N_{F/L}(F^\times)) = N_{F/L}(F^\times) \implies F' = F$. Since $F' \simeq F$, it is clear that F'/L is Galois with $\mathrm{Gal}(F/L) \simeq \mathrm{Gal}(F'/L)$. Hence F'/L is abelian. We claim that $\sigma(N_{F/L}(F^\times)) = N_{F'/L}(F'^\times)$. Recall the Artin map $\rho_L\colon L^\times \to \mathrm{Gal}(L_{\mathrm{ab}}/L)$ is such that $\ker(\rho_L|_{F'}) = N_{F'/L}(F'^\times)$. Let $\sigma(x) \in \sigma(F)$ and $y \in N_{F/L}(F^\times)$. Then

$$\begin{aligned} \rho_L(\sigma(y))(\sigma(x)) &= \sigma \cdot \rho_L(y) \cdot \sigma^{-1}(\sigma(x)) \\ &= \sigma \cdot \rho_L(y)(x) \\ &= \sigma(x) \end{aligned}$$

where the first equality is by [Iwa86, §6.3, Theorem 6.11]. Thus $\sigma(N_{F/L}(F^\times)) \subset N_{F'/L}(F'^\times)$. Conversely, if $x \in N_{F'/L}(F'^\times)$, then

$$\rho_L(x)|_{\sigma(F)} = \mathrm{Id} \implies \sigma \cdot \rho_L(\sigma^{-1}(x)) \cdot \sigma^{-1}|_{\sigma(F)} = \mathrm{Id} \implies \sigma^{-1}(x) \in N_{F/L}(F^\times)$$

so $x \in \sigma(N_{F/L}(F^\times))$. By our assumption, F and F' have the same norm groups so must in fact be equal by Proposition 1.13. Hence σ extends to a k-automorphism of F. $\square$

1.2 Weil-Deligne representations

One side of the local Langlands correspondence is phrased in terms of Weil-Deligne representations, which we introduce in this section. We use [Roh94, §1] as a reference for the content on the Weil group.

Let k be as before, and consider a Galois extension F/k (not necessarily finite). Let k' be the maximal unramified subextension so that $I_{F/k} = \mathrm{Gal}(F/k')$. Then there is a short exact sequence

$$1 \to I_{F/k} \to \mathrm{Gal}(F/k) \xrightarrow{\pi} \mathrm{Gal}(\mathfrak{f}_F \,/\, \mathfrak{f}_k) \to 1,$$

where π is the surjection to the Galois group of residue fields. Let $\mathrm{Fr}_{\mathfrak{f}_F \,/\, \mathfrak{f}_k} \in \mathrm{Gal}(\mathfrak{f}_F \,/\, \mathfrak{f}_k)$ be the Frobenius automorphism given by $x \mapsto x^q$.

Definition 1.16. The *Weil group* for F/k is $W(F/k) = \pi^{-1}\left(\langle \mathrm{Fr}_{\mathfrak{f}_F \,/\, \mathfrak{f}_k}\rangle\right) \subset \mathrm{Gal}(F/k)$.

In the case where $F = \overline{k}$ for a fixed choice of separable closure of k, let k^{nr} denote the maximal unramified subextension and $I_k = \mathrm{Gal}(\overline{k}/k^{\mathrm{nr}})$ the *inertia subgroup* of $G_k = \mathrm{Gal}(\overline{k}/k)$. Let k^{tame} be the maximal tamely ramified subextension of $\overline{k}/k$, and define the *wild inertia subgroup* $I_+ = \mathrm{Gal}(\overline{k}/k^{\mathrm{tame}})$. This is a normal subgroup of I_k. We define the *Weil group of* k to be $W_k = W(\overline{k}/k)$. Let $\mathrm{Fr} \in \mathrm{Gal}(\overline{k}/k)$

denote an element such that $\pi(\mathrm{Fr})$ is the Frobenius automorphism in $\mathrm{Gal}(\mathfrak{f}_{\overline{k}}\,/\,\mathfrak{f}_k)$. Then the short exact sequence

$$1 \to I_k \to W_k \xrightarrow{\pi} \langle \pi(\mathrm{Fr}) \rangle \to 1$$

is split due to the choice of Fr, so that $W_k = \langle \mathrm{Fr} \rangle \ltimes I_k$.

We endow W_k with the weakest topology such that I_k is an open subgroup, where I_k is equipped with its subspace topology inherited from G_k. We also require that left multiplication by Fr is a homeomorphism, making W_K a topological group. Then the identity in W_k has a neighbourhood basis consisting of open subgroups of I_k. The open subgroups of finite index in W_k are the subgroups $W_k \cap \mathrm{Gal}(\overline{k}/F)$, where F runs over finite extensions of k in $\overline{k}$.

Proposition 1.17. *The Weil group W_k satisfies the following properties*

1. *W_k is dense in G_k,*
2. *If F/k is a finite subextension, $W(\overline{k}/F) = W_k \cap \mathrm{Gal}(\overline{k}/F)$,*
3. *If F/k is a finite subextension, $W_k/W(\overline{k}/F) \simeq \mathrm{Gal}(F/k)$.*

Having defined the Weil group, we can now describe the image of the local Artin map.

Theorem 1.18 (Local Artin reciprocity contd.)**.** *The homomorphism $\rho_k\colon k^\times \to \mathrm{Gal}(k^{\mathrm{ab}}/k)$ as in Theorem 1.12 is a homeomorphism onto its image, which is $W(k^{\mathrm{ab}}/k) \simeq W_k^{\mathrm{ab}}$.*

To define a representation of W_k one needs to ensure that it is continuous.

Definition 1.19. A *representation of the Weil group* is a continuous homomorphism $\phi\colon W_k \to \mathrm{GL}(V)$, where V is a finite dimensional $\mathbb{C}$-vector space.

We say that ϕ is *unramified* if $\phi|_{I_k}$ is trivial, and *ramified* otherwise. We say that ϕ is *tamely ramified* if $\phi|_{I_+}$ is trivial, and *wildly ramified* otherwise.

Remark 1.20. The requirement that ϕ is continuous implies that ϕ must be trivial on an open subgroup of I_k (one argues this by using the fact that $\mathrm{GL}(V)$ has no nontrivial subgroups in a neighbourhood of the identity). In fact, a homomorphism $\phi\colon W_k \to \mathrm{GL}(V)$ is a representation if and only if it is trivial on an open subgroup of I_k. Since open subgroups of I_k have finite index, this implies that $\phi(I_k)$ is finite.

If V is one-dimensional, call ϕ a *character*. Then ϕ necessarily factors through W_k^{ab}, and we may identify ϕ as a character on $k^\times$ via the topological Artin isomorphism

$$k^\times \simeq W_k^{\mathrm{ab}}$$

as in Theorem 1.18. In this case, one usually normalizes ρ_k by requiring that $\rho_k(\varpi)|_{k^{\mathrm{nr}}} = \mathrm{Fr}^{-1}$, contrary to Theorem 1.12.

Example 1.21. Consider the character $\omega\colon W_k \to \mathbb{C}^\times$, with $\omega(I_k) = 1$ and $\omega(\mathrm{Fr}) = q^{-1}$. This corresponds to the character of $k^\times$ given by $x \mapsto |x|_k$. Indeed, $|\varpi|_k = q^{-1}$, and $|\mathcal{O}_k^\times| = 1$, where $\mathcal{O}_k^\times$ corresponds to I_k under the Artin map.

To define Weil-Deligne representations we need to consider a bigger group.

Definition 1.22. The *Weil-Deligne* group of k is $W_k \times \mathrm{SL}_2(\mathbb{C})$ equipped with the product topology.

Definition 1.23 (Weil-Deligne representations)**.** Let $\mathcal{G}$ be a complex Lie group with reductive identity component $\mathcal{G}^0$. A *Weil-Deligne representation* of $W_k \times \mathrm{SL}_2(\mathbb{C})$ is a $\mathcal{G}^0$-conjugacy class of homomorphisms $\varphi\colon W_k \times \mathrm{SL}_2(\mathbb{C}) \to \mathcal{G}$ such that $\varphi|_{I_k}$ is continuous, $\varphi|_{\mathrm{SL}_2(\mathbb{C})}$ is a homomorphism of algebraic groups over $\mathbb{C}$, and $\varphi(\mathrm{Fr})$ is semisimple.

We call a Weil-Deligne representation φ unramified/ramified or tamely ramified/wildly ramified etc. depending on the type of $\varphi|_{W_k}$.

Remark 1.24. This is the viewpoint of Weil-Deligne representations that is often used for the local Langlands correspondence. A Weil-Deligne representation can also be viewed as a pair (ϕ, N) where $\phi\colon W_k \to \mathrm{GL}(V)$ is a representation of W_k and N is a nilpotent endomorphism of V satisfying a certain compatibility condition with ϕ. For more details on Weil-Deligne representations more generally, one can consult [Roh94]. The equivalence of different definitions of Weil-Deligne representations is also discussed in [GR10, §2.1]

2 Langlands parameters

We can now introduce Langlands parameters, which in our case are Weil-Deligne representations that map to the dual group of a p-adic group. We use [GR10, §3] as a reference.

Let k be a finite extension of $\mathbb{Q}_p$. Consider a p-adic group $G(k)$ where G is a connected semisimple algebraic group over k. In addition we assume that G is split over k, i.e. that G has a maximal torus defined over k. Fix a maximal torus $T \subset G$ defined over k. Then G is determined up to isomorphism over k by its *root datum*

$$(X, \hat{X}, R, \hat{R})$$

where $X = X^*(T) = \mathrm{Hom}(T, \mathbb{G}_m)$ denotes the set of characters of T, $\hat{X} = X_*(T) = \mathrm{Hom}(\mathbb{G}_m, T)$ denotes the set of co-characters of T, R consists of the roots of T, and $\hat{R}$ the co-roots of T.

The dual root datum $(\hat{X}, X, \hat{R}, R)$ then defines a unique connected reductive group $\hat{G}$ over $\mathbb{C}$ known as the *dual group*, and we let $\hat{G} = \hat{G}(\mathbb{C})$. We take this group over $\mathbb{C}$ because later as part of the local Langlands correspondence we will compare these parameters to complex representations of G.

Example 2.1. The following table gives the dual group of some classical semisimple groups.

G	GL_n	SL_n	PGL_n	Sp_{2n}	SO_{2n+1}	SO_{2n}
$\hat{G}$	GL_n	PGL_n	SL_n	SO_{2n+1}	Sp_{2n}	SO_{2n}

Definition 2.2 (Langlands parameter)**.** Let G be a connected semisimple algebraic group over k. A Langlands parameter for G is a Weil-Deligne representation $\varphi\colon W_k \times \mathrm{SL}_2(\mathbb{C}) \to \hat{G}$ as in Definition 1.23.

We say two parameters φ, φ' are *equivalent* if they are $\hat{G}$-conjugate, that is there exists an element $g \in \hat{G}$ such that $\varphi(x) = g \cdot \varphi'(x) \cdot g^{-1}$ for all $x \in W_k \times \mathrm{SL}_2(\mathbb{C})$.

Definition 2.3 (Discrete Langlands parameter)**.** Let A_φ denote the centralizer of $\varphi(W_k \times \mathrm{SL}_2(\mathbb{C}))$ in $\hat{G}$. Then φ is *discrete* if A_φ is finite.

Remark 2.4. For general groups, a Langlands parameter maps to the *Langlands dual group* ${}^LG$ of G. However because we have assumed that G is split over k, we may take ${}^LG = \hat{G}$.

Remark 2.5 (Discrete parameters have finite image on W_k)**.** As described in Remark 1.20, if φ is a Langlands parameter then $\varphi(I_k)$ is finite. Since $\varphi(\mathrm{Fr})$ normalizes $\varphi(I_k)$, some power $\varphi(\mathrm{Fr})^n$ centralizes $\varphi(I_k)$. Thus if A_φ is finite it follows that $\varphi(\mathrm{Fr})$ has finite order, and so $\varphi(W_k)$ is finite. The structure of W_k then implies that $\varphi(W_k) \simeq \mathrm{Gal}(F/k)$, where F/k is a finite Galois extension. Explicitly $F = (\overline{k})^{\ker(\varphi|_{W_k})}$. In particular $\varphi(I_k) = G_0(F/k)$ and $\varphi(I_k^+) = G_1(F/k)$, so the ramification of φ depends on that of F/k.

Example 2.6 (Principal parameter, [GR10, §3.3])**.** There is an important discrete parameter called the principal parameter, denoted by φ_0. This parameter is such that $\varphi_0|_{W_k}$ is trivial, and $\varphi_0((\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}))$ is unipotent in $\hat{G}$. It is obtained from a choice of pinning in $\hat{G}$. This has centralizer equal to the center $\hat{Z}$ of $\hat{G}$ and appears in the statement of the formal degree conjecture.

2.1 Simple wild parameters

Suppose that $\varphi\colon W_k \times \mathrm{SL}_2(\mathbb{C}) \to \hat{G}$ is a discrete Langlands parameter for $G(k)$. Consider the adjoint representation $\mathrm{Ad}\colon \hat{G} \to \mathrm{GL}(\hat{\mathfrak{g}})$ of $\hat{G}$ on its Lie algebra $\hat{\mathfrak{g}}$ and take the composite

$$W_k \times \mathrm{SL}_2(\mathbb{C}) \xrightarrow{\varphi} \hat{G} \xrightarrow{\mathrm{Ad}} \mathrm{GL}(\hat{\mathfrak{g}}).$$

Let $\hat{\mathfrak{g}}_N$ denote the subspace of $\hat{\mathfrak{g}}$ fixed by $\mathrm{Ad}(\varphi((\begin{smallmatrix}1&1\\0&1\end{smallmatrix})))$. As $\varphi(W_k)$ commutes with $\varphi(\mathrm{SL}_2(\mathbb{C}))$, $\hat{\mathfrak{g}}_N$ is invariant under the action of W_k. We let $\hat{\mathfrak{g}}_N^I$ denote the subspace of $\hat{\mathfrak{g}}_N$ fixed by I_k. Since I_k is normal in W_k, $\hat{\mathfrak{g}}_N^I$ is preserved by the action of $\varphi(\mathrm{Fr})$. We define the L-function of φ as the determinant of an operator on $\hat{\mathfrak{g}}_N^I$.

Definition 2.7. The *adjoint L-function* of a discrete parameter φ is given by

$$L(\varphi, \hat{\mathfrak{g}}, s) = \det(\mathrm{I} - q^{-s}\varphi(\mathrm{Fr}) \mid \hat{\mathfrak{g}}_N^I).$$

We also introduce the adjoint gamma value of a discrete parameter, which appears in the formal degree conjecture in §5.

Definition 2.8. The *adjoint gamma value* of a discrete parameter φ is given by

$$\gamma(\varphi) = \frac{L(\varphi, \hat{\mathfrak{g}}, 1) \cdot \epsilon(\varphi, \hat{\mathfrak{g}}, 0)}{L(\varphi, \hat{\mathfrak{g}}, 0)},$$

where $\epsilon(\varphi, \hat{\mathfrak{g}}, s)$ is the epsilon factor associated to the representation $\mathrm{Ad} \circ \varphi$ (see [GR10, §2.2]).

Note that $\mathrm{Ad} \circ \varphi$ defines a finite-dimensional complex representation of $H = \mathrm{Gal}(F/k)$, where $F = (\overline{k})^{\ker \varphi|_{W_k}}$. Thus one can define the Swan conductor of this representation of H as in Definition 1.9, which we denote by $\mathrm{sw}(\varphi)$. This is known as the *adjoint swan conductor*. Note that equivalent Langlands parameters give rise to the same adjoint Swan conductor.

Example 2.9. If φ is tamely ramified, i.e. $\varphi(I_+) = G_1(F/k) = \{1\}$, then $\mathrm{sw}(\varphi) = 0$.

In [GR10, §5], Gross and Reeder give a conjectural inequality involving the adjoint Swan conductor of a discrete Langlands parameter. This provides a lower-bound for the adjoint Swan conductor, and they focus on when this bound is sharp. This leads them to the definition of a simple wild parameter.

Definition 2.10. A discrete parameter $\varphi\colon W_k \times \mathrm{SL}_2(\mathbb{C}) \to \hat{G}$ with $F = (\overline{k})^{\ker \phi|_{W_k}}$ is said to be *inertially discrete* if $\hat{\mathfrak{g}}^{\varphi(I_k)} = \hat{\mathfrak{g}}^{G_0(F/k)} = \{1\}$.

Lemma 2.11 ([GR10, Lemma 5.3])**.** *Let $\varphi\colon W_k \times \mathrm{SL}_2(\mathbb{C}) \to \hat{G}$ be a discrete Langlands parameter. Then $\hat{\mathfrak{g}}^{\varphi(I_k)} = \{1\}$ if and only if $L(\varphi, \hat{\mathfrak{g}}, s) = 1$. If this is the case, then $\varphi(\mathrm{SL}_2(\mathbb{C})) = 1$.*

Conjecture 2.12 ([GR10, Conjecture 5.2])**.** *Let $\varphi\colon W_k \times \mathrm{SL}_2(\mathbb{C}) \to \hat{G}$ be an inertially discrete Langlands parameter. Then $\mathrm{sw}(\varphi) \geq \mathrm{rank}\, \hat{\mathfrak{g}}$.*

Definition 2.13 (Simple wild parameter)**.** A discrete Langlands parameter $\varphi\colon W_k \times \mathrm{SL}_2(\mathbb{C}) \to \hat{G}$ is called a *simple wild parameter* if $\hat{\mathfrak{g}}^{\varphi(I_k)} = \{1\}$ and $\mathrm{sw}(\varphi) = \mathrm{rank}\, \hat{\mathfrak{g}}$.

Gross and Reeder obtain a rather explicit description of simple wild parameters in the following case.

Proposition 2.14 ([GR10, Proposition 5.6])**.** *Assume that $\varphi\colon W_k \times \mathrm{SL}_2(\mathbb{C}) \to \hat{G}$ is an inertially discrete parameter satisfying $\mathrm{sw}(\varphi) = \mathrm{rank}\, \hat{\mathfrak{g}}$, and that the residue characteristic p of k does not divide the order of the Weyl group of $\hat{G}$. Let $H = \mathrm{Gal}(F/k)$ where $F = (\overline{k})^{\ker \varphi|_{W_k}}$ and let $G_0 \geq G_1 \geq \cdots \geq G_{c+1} = \{1\}$ be the lower ramification groups of H. Then*

1. *$\varphi(G_1)$ lies in a unique maximal torus $\hat{T}$ of $\hat{G}$ and $\varphi(H)$ lies in the normalizer $N(\hat{T})$ of $\hat{T}$ in $\hat{G}$,*

2. *The image of G_0/G_1 in $N(\hat{T})/\hat{T}$ is generated by a Coxeter element of order h, where h is the Coxeter number of $\hat{G}$,*
3. *$G_2 = 1$ and G_1 is an elementary abelian p-abelian group of order p^a, where a is the order of p in the group $(\mathbb{Z}/h\mathbb{Z})^\times$,*
4. *H has upper ramification breaks at 0 and $1/h$.*

Conversely, a parameter satisfying these properties is a simple wild parameter.

2.2 Simple wild parameters for $\mathrm{SL}_2(\mathbb{Q}_p)$

In this section we compute all simple wild parameters for $G = \mathrm{SL}_2$ over $k = \mathbb{Q}_p$. In this case $\hat{G} = \mathrm{PGL}_2(\mathbb{C})$ is a split, connected, reductive group of rank 1. Since the conjectural minimal non-zero value of the swan conductor of a discrete parameter is equal to the rank of $\hat{G}$, our goal is to classify all inertially discrete Langlands parameters $\varphi\colon W_k \times \mathrm{SL}_2(\mathbb{C}) \to \hat{G}$ such that the image of $\mathrm{SL}_2(\mathbb{C})$ is trivial, and $\mathrm{sw}(\varphi) = 1$.

Firstly, $\varphi(W_k)$ will be a finite subgroup of $\hat{G}$. The following result, due to Klein, classifies the finite subgroups of $\mathrm{PGL}_2(\mathbb{C})$.

Proposition 2.15. *[Kle56] The finite subgroups of $\mathrm{PGL}_2(\mathbb{C})$ are, up to isomorphism,*

- *the cyclic groups C_n,*
- *the dihedral groups D_n (with $2n$ elements),*
- *the tetrahedral group A_4,*
- *the symmetric group S_4,*
- *the alternating group A_5.*

Moreover, up to conjugation, all of these groups appear as subgroups of $\mathrm{PGL}_2(\mathbb{C})$ exactly once.

Since the Swan conductor is defined up to equivalence of Langlands parameters, and these finite subgroups of $\mathrm{PGL}_2(\mathbb{C})$ appear only once up to conjugation, we may explicitly choose generators of a copy of any of these groups in $\mathrm{PGL}_2(\mathbb{C})$.

Note that C_n cannot be the image of a discrete parameter. Indeed we can take

$$C_n = \left\langle \begin{pmatrix} \zeta_n & 0 \\ 0 & 1 \end{pmatrix} \right\rangle \subset \mathrm{PGL}_2(\mathbb{C})$$

where ζ_n is a primitive n-th root of unity. This subgroup has centralizer equal to all diagonal matrices, which in particular is not finite. In addition, as A_5 is not solvable, it cannot be the Galois group of an extension of $\mathbb{Q}_p$, hence cannot arise as the image of a discrete parameter either.

To determine the simple wild parameters, we first consider $p = 2$, and then consider odd primes.

2.2.1 Parameters for $\mathrm{SL}_2(\mathbb{Q}_2)$

A field extension of $\mathbb{Q}_p$ with Galois group isomorphic to S_4 can only occur if $p = 2$. Indeed in this case G_0, a normal subgroup of S_4, is one of C_1, C_2^2, A_4, S_4. Then G_1 is a p-Sylow subgroup of G_0, and normal in G_0 which forces $p = 2$ (by observing that the normal Sylow subgroups of all possible G_0 are only 2-Sylow subgroups). A similar argument for $\mathrm{Gal}(F/k) \simeq A_4$ shows that $k = \mathbb{Q}_2$ also.

Suppose $\varphi(W_k) \simeq A_4$. In [Wei74], Weil determines that there is a unique extension of $\mathbb{Q}_2$ with Galois group isomorphic to A_4. This is obtained as the splitting field of $x^4 + 2x^3 + 2x^2 + 2$, with ramification filtration

$$A_4 \supset G_0 = C_2 \times C_2 \supset G_1 = C_2 \times C_2 \supset G_2 = \{1\}.$$

Consider $A_4 = \langle g_1, g_2, g_3 \mid g_1^2 = g_2^2 = g_3^3 = 1,\ g_3 g_1 g_3^{-1} = g_2,\ g_3 g_2 g_3^{-1} = g_1 g_2 = g_2 g_1 \rangle$ as a subgroup of $\mathrm{PGL}_2(\mathbb{C})$ by setting

$$g_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad g_3 = \begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix}.$$

This subgroup has finite stabilizer in $\mathrm{PGL}_2(\mathbb{C})$. One can compute that $C_{\hat{G}}(g_1) \cap C_{\hat{G}}(g_2) = \langle g_1, g_2 \rangle \simeq C_2 \times C_2$. Since none of these elements centralize g_3, one has $C_{\hat{G}}(A_4) = \{\mathrm{I}_2\}$ is of size one.

Consider the adjoint action of A_4 on $\hat{\mathfrak{g}} = \mathfrak{sl}_2(\mathbb{C})$. Take a basis

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

of $\mathfrak{sl}_2(\mathbb{C})$. Then

$$\mathrm{Ad}(g_1) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathrm{Ad}(g_2) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \mathrm{Ad}(g_3) = \frac{i}{2} \begin{pmatrix} 1 & 1 & -\frac{2}{i} \\ -1 & -1 & -\frac{2}{i} \\ 1 & -1 & 0 \end{pmatrix}.$$

As $\hat{\mathfrak{g}}^{G_0} = \hat{\mathfrak{g}}^{G_1} = 0$, one computes that $\mathrm{sw}(\varphi) = 3$, which is not minimal.

Now suppose that $\varphi(W_k) \simeq S_4$. Again we follow [Wei74] to describe explicit extensions of $\mathbb{Q}_2$ with Galois group S_4. There is a unique extension of $\mathbb{Q}_2$ with Galois group S_3, given by $L = \mathbb{Q}_2(\zeta_3, \sqrt[3]{2})$. Note $C_2 \times C_2 \trianglelefteq S_4$ with quotient group S_3. It follows that any extension K of $\mathbb{Q}_2$ with Galois group S_4 must contain L as a subfield. In particular, K is a biquadratic extension of L . There are three of these, up to isomorphism. These are described in [Wei74], and the lower ramification groups are computed using SageMath [The24]. They are

1. $K_1 = L(\epsilon, \epsilon')$ where $\epsilon^2 = 3(1 + \sqrt[3]{2})$ and $\epsilon'^2 = 3(1 + \zeta_3 \sqrt[3]{2})$. Then $\mathrm{Gal}(K_1/\mathbb{Q}_2) = S_4$. One has $G_0 = A_4$, $G_1 = G_2 = G_3 = G_4 = G_5 = C_2 \times C_2$, and $G_6 = 1$.

2. $K_2 = L(\epsilon, \epsilon')$, where $\epsilon^2 = 1 + (\sqrt[3]{2})^2$ and $\epsilon'^2 = 1 + \zeta_3^2 (\sqrt[3]{2})^2$. Then $\mathrm{Gal}(K_2/\mathbb{Q}_2) = S_4$. One has $G_0 = A_4$, $G_1 = C_2 \times C_2$, and $G_2 = 1$.

3. $K_3 = L(\epsilon, \epsilon')$ where $\epsilon^2 = 3(1 + \sqrt[3]{2})(1 + (\sqrt[3]{2})^2)$ and $\epsilon'^2 = 3(1 + \zeta_3 \sqrt[3]{2})(1 + \zeta_3^2 (\sqrt[3]{2})^2)$. This has the same lower ramification groups as $K_1/\mathbb{Q}_2$.

Since the Swan conductor of a Langlands parameter which factors through the Galois group of one of these extensions only depends on the image of inertia which is A_4, we can use the computations above. One calculates that for a parameter φ that factors through $\mathrm{Gal}(K_1/\mathbb{Q}_2)$ or $\mathrm{Gal}(K_3/\mathbb{Q}_2)$, $\mathrm{sw}(\varphi) = 5$ is not minimal.

If φ factors through $\mathrm{Gal}(K_2/\mathbb{Q}_2)$ then $\mathrm{sw}(\varphi) = 1$, hence this is a simple wild parameter. Let $S_4 \le \mathrm{PGL}_2(\mathbb{C})$ be a fixed copy of the symmetric group. Then one obtains a new parameter by setting $\varphi' = \sigma \circ \varphi$ where σ is an automorphism of S_4. If σ does not arise as conjugation by some matrix in $\mathrm{PGL}_2(\mathbb{C})$, then φ, φ' are distinct non-isomorphic simple wild parameters. But all automorphisms of S_4 are inner, so there is a unique simple wild parameter up to equivalence for $\mathrm{SL}_2(\mathbb{Q}_2)$ with image isomorphic to S_4.

2.2.2 Parameters with dihedral image

Now we study the case where $\varphi(W_k) \simeq D_n$. Observe that for odd primes, this is the only case where discrete parameters may occur. We fix the following copy of $D_n \le \mathrm{PGL}_2(\mathbb{C})$.

$$D_n = \left\langle \sigma = \begin{pmatrix} \zeta_n & 0 \\ 0 & 1 \end{pmatrix}, \quad \tau = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle \le \mathrm{PGL}_2(\mathbb{C}).$$

The centralizer of D_2 is D_2 and for $n > 2$ one has $C_{\hat{G}}(D_n) = \langle \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \rangle$ of size 2. Let $\{e, f, h\}$ be the basis for $\hat{\mathfrak{g}}$ as before. Then

$$\mathrm{Ad}(\sigma) = \begin{pmatrix} \zeta_n & 0 & 0 \\ 0 & \zeta_n^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathrm{Ad}(\tau) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Thus $\hat{\mathfrak{g}}^{C_n} = \langle h \rangle$ and $\hat{\mathfrak{g}}^{\tau} = \langle e + f \rangle$.

Suppose that $F/\mathbb{Q}_p$ has Galois group $\mathrm{Gal}(F/\mathbb{Q}_p) \simeq D_n$. The only cyclic quotients of D_n are C_1, C_2. Since G_{-1}/G_0 is cyclic, this forces $G_0 = C_n$, or $G_0 = D_n$, or $G_0 = D_{n/2}$ when n is even (and > 2). First, we show that the image of inertia cannot be cyclic.

Lemma 2.16. *Suppose that $\varphi\colon W_k \to \hat{G}$ is a discrete Langlands parameter such that W_k factors through $\mathrm{Gal}(F/k) \simeq D_n$, where F/k has inertia subgroup C_n, and F/k is wildly ramified. Then $\mathrm{sw}(\varphi) \geq 2$ is not minimal.*

Proof. Since F/k is wildly ramified, $G_1 \neq \{1\}$ is the unique p-Sylow subgroup of G_0. It follows that $n = p^j m$ for some $j \geq 1$ and m coprime to p. Therefore

$$G_{-1} = D_{p^j m}, \quad G_0 = C_{p^j m}, \quad G_1 = C_{p^j}, \quad G_i = C_{p^{j_i}},\ i \geq 1,$$

where j_i are integers with $j_1 = j$, $j_i \leq j$. For $i \geq 1$, G_i/G_{i+1} is an elementary abelian p-group, hence either C_1 or C_p. For $i \geq 1$, if G_i is non-trivial, one has $\dim \hat{\mathfrak{g}}^{G_i} = 1$. Therefore

$$\mathrm{sw}(\varphi) = \sum_{i=1}^{c} \frac{2}{[G_0\colon G_i]} = 2\sum_{i=1}^{c} \frac{p^{j_i}}{p^j m} = \frac{2}{mp^j}\left(\sum_{i=1}^{c} p^{j_i}\right).$$

Let a_i be the number of ramification groups equal to C_{p^i} for $i = 1, \dots j$. Then to have a minimal Swan conductor of 1 we need

$$1 = \mathrm{sw}(\varphi) = \frac{2}{mp^j} \cdot \left(\sum_{i=1}^{j} a_i p^i\right).$$

Let L be the quadratic unramified subextension of F/k. Then by Proposition 1.3 one has $\mathrm{Gal}(F/L) = G_0(F/L) = C_{p^j m}$, $G_i(F/L) = C_{p^{j_i}}$. In particular the lower ramification groups of F/L and F/k change at the same indices. Since F/L is cyclic, by Theorem 1.8 one has $G_i(F/L) \neq G_{i+1}(F/L) \implies \phi_{F/L}(i)$ is an integer. Thus $\phi_{F/L}(a_j)$, $\phi_{F/L}(a_j + a_{j-1}), \dots$ are all integers. In particular,

$$\phi_{F/L}(a_j + a_{j-1} + \cdots + a_1) = \frac{1}{p^j m}(a_j p^j + \cdots + a_1 p) \in \mathbb{Z} \implies \sum_{i=1}^{j} a_i p^i \equiv 0 \pmod{mp^j}.$$

Hence $\mathrm{sw}(\varphi) \geq 2$ cannot be minimal. □

We also show that we cannot have a Langlands parameter of dihedral image when $k = \mathbb{Q}_2$.

Lemma 2.17. *There is no simple wild parameter $\varphi\colon W_k \to \hat{G}$ with $\varphi(W_k) = D_n$ and $k = \mathbb{Q}_2$.*

Proof. Such a Langlands parameter needs to be wildly ramified. We showed above that G_0 cannot be cyclic. Thus G_0 is dihedral, and G_1 is the unique 2-Sylow subgroup with G_0/G_1 cyclic of order prime to 2. This implies that $G_1 = G_0$ is a 2-group. Then since $\hat{\mathfrak{g}}^{D_n} = 0$ for all $n \geq 2$, one has

$$\mathrm{sw}(\varphi) = \sum_{i=1}^{c} \frac{3}{[D_0 : D_i]} = 3 + \sum_{i=2}^{c} \frac{3}{[D_0\colon D_i]}$$

which cannot be equal to 1. □

Assume from now on that $k = \mathbb{Q}_p$ with p odd. By Lemma 2.16, one must have that $\varphi(I_k)$ is dihedral and so $\hat{\mathfrak{g}}^{\varphi(I_k)} = 0$. Therefore φ is inertially discrete. The order of the Weyl group of $\hat{G}$ is two, and so we can use Proposition 2.14 to force some properties of such discrete parameters with minimal Swan conductor. In this case for an inertially discrete parameter to have minimal Swan conductor, it must factor through $\mathrm{Gal}(F/k)$ where $G_0(F/k)/G_1(F/k) = C_2$, $G_1(F/k) = C_p$ and $G_2(F/k) = C_1$. Thus one of the following two cases to occur:

$$\begin{array}{lllll} (\mathrm{I}): & G_{-1} = D_p, & G_0 = D_p, & G_1 = C_p, & G_2 = 1, \\ (\mathrm{II}): & G_{-1} = D_{2p}, & G_0 = D_p, & G_1 = C_p, & G_2 = 1. \end{array}$$

If such a parameter φ exists, then $\mathrm{sw}(\varphi) = \frac{1}{2}(3-1) = 1$ is minimal. Thus we need to describe all extensions with these Galois group and ramification groups.

In either case, there is a subfield L such that $\mathrm{Gal}(F/L) = C_p$ is totally ramified. Moreover, since $G_i(F/L) = G_i(F/\mathbb{Q}_p) \cap \mathrm{Gal}(F/L)$ by Proposition 1.3 one obtains that

$$G_0(F/L) = G^0(F/L) = C_p, \quad G_1(F/L) = G^1(F/L) = C_p, \quad G_2(F/L) = G^2(F/L) = C_1.$$

as in Example 1.6. First we count the number of such extensions.

Proposition 2.18. *Let $L/\mathbb{Q}_p$ be a finite extension. There are p totally ramified extensions F/L such that* $\mathrm{Gal}(F/L) = G^0(F/L) = G^1(F/L) = C_p$, $G^2(F/L) = C_1$.

Proof. Let β be a choice of uniformizer for L, and suppose that the size of the residue field of L is q. Write

$$L^\times \simeq \langle\beta\rangle \times \mu_{q-1} \times (1 + \mathfrak{P}),$$

where μ_{q-1} is the group of $(q-1)$-th roots of unity in L, and $\mathfrak{P}$ is the maximal ideal of $\mathcal{O}_L$.

By Proposition 1.13, the degree p extensions of F/L are in one-to-one correspondence with index p subgroups of $L^\times$. Such an index p subgroup contains $L^{\times p} = \langle\beta\rangle^p \times \mu_{q-1} \times (1+\mathfrak{P})^p$. In addition, if F/L has the ramification groups as in the proposition, then Theorem 1.14 implies that

$$\rho_{F/L}(\mathcal{O}_L^\times) = C_p, \quad \rho_{F/L}(1+\mathfrak{P}) = C_p, \quad \rho_{F/L}(1+\mathfrak{P}^2) = 1.$$

In particular, $1+\mathfrak{P}^2 \subset N_{F/L}(F^\times)$, and $1+\mathfrak{P} \not\subset N_{F/L}(F^\times)$. Therefore one has $A = \langle\beta\rangle^p \times \mu_{q-1} \times (1+\mathfrak{P}^2) \subset N_{F/L}(F^\times)$ and so $L^\times/N_{F/L}(F^\times)$ is a quotient of $L^\times/A$ and extensions correspond to index p subgroups of $L^\times/A$. We have

$$L^\times/A \simeq \frac{\langle\beta\rangle}{\langle\beta\rangle^p} \times \frac{(1+\mathfrak{P})}{(1+\mathfrak{P}^2)} = \langle\overline{\beta}\rangle \times \langle\overline{a}\rangle = C_p \times C_q$$

where we use $\overline{\cdot}$ to denote reduction mod A, and $a \in 1+\mathfrak{P}$ generates the quotient $(1+\mathfrak{P})/(1+\mathfrak{P}^2) \simeq \mathbb{F}_q$. There are $p+1$ index p subgroups of the form $\langle\overline{\beta^i a}\rangle$ for $i = 0, \ldots p-1$ and $\langle\overline{\beta}\rangle \times \langle\overline{a^p}\rangle$. One cannot have $\overline{a} \in N_{F/L}(F^\times)/A$, as this implies $1+\mathfrak{P} \subset N_{F/L}(F^\times)$. Thus the subgroup $\langle\overline{a}\rangle \subset L^\times/A$ cannot correspond to an extension, but the other p subgroups correspond to extensions of L satisfying the conditions of the proposition. $\square$

Assume we are in Case (I), so that $\mathrm{Gal}(F/k) \simeq D_p$. Let L be the quadratic ramified subextension with $\mathrm{Gal}(F/L) = C_p$. There are 2 possible choices for L up to isomorphism. We show that there is a unique extension F/L with the required ramification groups and such that $F/\mathbb{Q}_p$ is Galois.

Proposition 2.19. *Let p be an odd prime. Let $L/\mathbb{Q}_p$ be a quadratic ramified extension. Then there is a unique extension F/L such that*

1. *F/L is Galois with* $\mathrm{Gal}(F/L) \simeq C_p$,
2. *F/L has ramification groups $G_0(F/L) = G_1(F/L) = C_p$, $G_2(F/L) = C_1$,*

3. $F/\mathbb{Q}_p$ is Galois.

This unique extension has $\mathrm{Gal}(F/\mathbb{Q}_p) = G_0(F/\mathbb{Q}_p) = D_p$*,* $G_1(F/\mathbb{Q}_p) = C_p$*,* $G_2(F/\mathbb{Q}_p) = C_1$*.*

Proof. Recall the notation in the proof of Proposition 2.18. There are p extensions that satisfy the second two conditions of the proposition, whose norm groups map to $\langle\overline{\beta^i a}\rangle$, $i = 1, \dots p-1$ and $\langle\overline{\beta}\rangle \times \langle\overline{a^p}\rangle = \langle\overline{\beta}\rangle$ in $L^\times/A$, where $A = \langle\beta\rangle^p \times \mu_{p-1} \times (1 + \mathfrak{P}^2)$. Here our choice of β is such that $L = \mathbb{Q}_p(\beta)$ and $\mathrm{Gal}(L/\mathbb{Q}_p) = \langle\tau\rangle$ where $\tau(\beta) = -\beta$.

By Lemma 2.20, if $\mathrm{Gal}(L/\mathbb{Q}_p)$ preserves $N_{F/L}(F^\times)$ then $F/\mathbb{Q}_p$ is Galois. Write $a = 1 + u$ where $u \in \mathfrak{P}$. Then $\tau(a) = 1 - u$ and $a \cdot \tau(a) = 1 - u^2 \in N_{F/L}(F^\times)$, so that $\tau(a) \equiv a^{-1} \mod N_{F/L}(F^\times)$. Suppose that $N_{F/L}(F^\times)/A = \langle\overline{\beta^i a}\rangle$ for $i = 1, \dots p = 1$. Then

$$\tau(\beta^i a) = \tau(\beta)^i \tau(a) \equiv \beta^i a^{-1} \mod N_{F/L}(F^\times),$$

since $-1 \in N_{F/L}(F^\times)$. If $\beta^i a^{-1} \in N_{F/L}(F^\times)$, then $\beta^i a \cdot (\beta^i a^{-1})^{-1} = a^2 \in N_{F/L}(F^\times)$. But then $\overline{a} \in \langle\overline{a^2}\rangle \subset N_{F/L}(F^\times)/A$ and so $a \in N_{F/L}(F^\times)$, a contradiction. Thus $\tau(\beta a^i) \notin N_{F/L}(F^\times)$ and so $F/\mathbb{Q}_p$ is not Galois for $N_{F/L}(F^\times)/A = \langle\overline{\beta^i a}\rangle$.

On the other hand if $N_{F/L}(F^\times) = \langle\beta\rangle \times \mu_{p-1} \times (1 + \mathfrak{P})$ (corresponding to $N_{F/L}(L^\times)/A = \langle\overline{\beta}\rangle$) then it is clearly preserved by τ. Therefore this norm group corresponds to the unique extension of L that satisfies the properties of the proposition.

Finally, we compute the lower ramification groups of $F/\mathbb{Q}_p$. The extension $F/\mathbb{Q}_p$ is Galois of order $2p$, hence is either abelian or D_p. Note that a generates $L^\times/N_{F/L}(F^\times)$, and by above $\mathrm{Gal}(L/k)$ acts on this quotient by inversion. Hence it acts non-trivially (acting by conjugation) on $\mathrm{Gal}(F/L)$. Thus $\mathrm{Gal}(F/\mathbb{Q}_p) = D_p$. Since $L/\mathbb{Q}_p$ and F/L are totally ramified, so is $F/\mathbb{Q}_p$, and $G_0 = D_p$. The tame quotient G_0/G_1 is cyclic of order prime to p, hence $G_1 = C_p$. Since $1 = G_2(F/L) = G_2(F/\mathbb{Q}_p) \cap C_p$, it follows that $G_2(F/\mathbb{Q}_p)$ must be trivial. □

Therefore any $F/\mathbb{Q}_p$ as in Case (I) is unique up to choice of the quadratic ramified subfield. Since there are two quadratic ramified extensions of $\mathbb{Q}_p$ for p odd, we obtain that there are two extensions of $\mathbb{Q}_p$ with Galois group D_p and required ramification groups.

Thus there are two Galois extensions through which φ can factor so that it has image isomorphic to D_p and is a simple wild parameter. To count the total number of non-equivalent simple wild parameters with image isomorphic to D_p, we multiply 2 by the number of automorphisms of our fixed $D_p \leq \mathrm{PGL}_2(\mathbb{C})$ that do not emerge as conjugation by an element of $\mathrm{PGL}_2(\mathbb{C})$. The following lemma thus shows that there are $2 \cdot \frac{p-1}{2} = p-1$ simple wild parameters φ up to equivalence with $\varphi(W_k) \simeq D_p$.

Lemma 2.20. *Let $\mathcal{D}$ be the group of automorphisms of $D_p \leq \hat{G}$ that emerge as conjugation by an element of $\hat{G}$. Then* $\mathrm{Aut}(D_p)/\mathcal{D}$ *has order $\frac{p-1}{2}$.*

Proof. Recall $D_p = \langle\sigma, \tau | \sigma^p = \tau^2 = 1, \tau\sigma\tau\sigma = 1\rangle$. Then $\mathrm{Aut}(D_p) = C_p \ltimes (C_p)^\times$ where $(a, b) \in \mathrm{Aut}(D_p)$ is defined by $\tau \mapsto \tau\sigma^a$, $\sigma \mapsto \sigma^b$. Conjugation by the element $\tau^i\sigma^j$ is an automorphism of D_p that sends

$$\tau \mapsto \tau\sigma^{2j}, \quad \sigma \mapsto \begin{cases} \sigma & i = 0, \\ \sigma^{-1} & i = 1 \end{cases}.$$

Thus, since multiplication by 2 is injective on C_p, there are $2p$ distinct inner automorphisms of D_p and so $\mathrm{Out}(D_p)$ has order $\frac{p-1}{2}$ and is cyclic. It is generated by the automorphism $\tau \mapsto \tau$, $\sigma \mapsto \sigma^\alpha$, for α a generator of $(C_p)^\times/\{\pm 1\}$. We show that this automorphism cannot be obtained by conjugating by some $g \in \mathrm{PGL}_2(\mathbb{C})$. Indeed, if g fixes τ then $g = \left(\begin{smallmatrix} a & b \\ b & a \end{smallmatrix}\right)$ or $g = \left(\begin{smallmatrix} a & b \\ -b & -a \end{smallmatrix}\right)$. Then

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix} \begin{pmatrix} \zeta_p & 0 \\ 0 & 1 \end{pmatrix} = \lambda \begin{pmatrix} \zeta_p^\alpha & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ b & a \end{pmatrix}, \quad \lambda \in \mathbb{C}^\times \implies$$

$$\begin{cases} a(\lambda\zeta_p - \zeta_p) & = 0, \\ b(\lambda\zeta_p^{\alpha} - 1) & = 0, \\ b(\lambda - \zeta_p) & = 0, \\ a(\lambda - 1) & = 0, \end{cases} \implies a = b = 0 \quad \Rightarrow\!\Leftarrow,$$

and we reach a similar conclusion taking $g = \left(\begin{smallmatrix} a & b \\ -b & -a \end{smallmatrix}\right)$. $\square$

Now we consider Case (II), where $\mathrm{Gal}(F/\mathbb{Q}_p) = D_{2p}$. There is a unique normal C_2 subgroup of D_{2p} with $D_{2p}/C_2 \simeq D_p$. Hence there is a unique subextension $k \subset F' \subset F$ such that $\mathrm{Gal}(F/F') = C_2$ and $\mathrm{Gal}(F'/\mathbb{Q}_p) = D_p$. We claim that F is determined by F'.

Proposition 2.21. *If $F/\mathbb{Q}_p$ is an extension satisfying the conditions of Case* (II) *then $F = F'(\zeta_{p^2-1})$ where $F'/\mathbb{Q}_p$ is a Galois extension satisfying the conditions of Case* (I)*.*

Proof. If $F/\mathbb{Q}_p$ is as in Case (II), then one can compute that the upper ramification groups have breaks at 0 and 1/2, with $G^0(F/\mathbb{Q}_p) = D_p$, $G^{1/2}(F/\mathbb{Q}_p) = C_p$, and $G^{1/2+\epsilon}(F/\mathbb{Q}_p) = C_1$ for $\epsilon > 0$. Let F' be the unique subextension of $F/\mathbb{Q}_p$ with $\mathrm{Gal}(F/F') = C_2 \triangleleft D_{2p}$. By Proposition 1.7, $G^s(F'/\mathbb{Q}_p) = G^s(F/\mathbb{Q}_p)C_2/C_2$ and therefore the upper ramification groups of $F'/\mathbb{Q}_p$ have breaks at 0 and 1/2 also with $G^0(F'/\mathbb{Q}_p) = D_p$, $G^1(F'/\mathbb{Q}_p) = C_p$ and $G^{1/2+\epsilon}(F'/\mathbb{Q}_p) = C_1$ for $\epsilon > 0$.

One has $G_0(F/F') = \mathrm{Gal}(F/F') \cap G_0(F/\mathbb{Q}_p) = C_2 \cap D_p = C_1$. Thus F/F' is unramified, and $F'/\mathbb{Q}_p$ is totally ramified. Therefore $G_0(F'/\mathbb{Q}_p) = \mathrm{Gal}(F'/\mathbb{Q}_p) = D_p$. Since $G_0(F'/\mathbb{Q}_p)/G_1(F'/\mathbb{Q}_p)$ is cyclic of order prime to p, $G_1(F'/\mathbb{Q}_p)$ is either D_p or C_p. However the first case cannot occur, since then some higher quotient $G_i(F'/\mathbb{Q}_p)/G_{i+1}(F'/\mathbb{Q}_p)$ for $i \geq 1$ would not be an elementary p-group (as required by Proposition 1.3). Therefore $G_1(F'/\mathbb{Q}_p) = C_p$. Then $G_2(F'/\mathbb{Q}_p) = G^{\phi_{F'/\mathbb{Q}_p}(s)}(F'/\mathbb{Q}_p)$ with $\phi_{F'/\mathbb{Q}_p}(2) = \frac{1}{|G_0|}(|G_1| + |G_2|) = \frac{1}{2p}(p + |G_2|) > \frac{1}{2}$ and so $G_2(F'/\mathbb{Q}_p) = C_1$.

Therefore $F'/\mathbb{Q}_p$ is an extension as in Case (I). Moreover, since F/F' is a degree 2 unramified extension, and the size of the residue field of F' is p, it follows that $F = F'(\zeta_{p^2-1})$. $\square$

Thus an extension satisfying the conditions of Case (II) is determined by its unique subextension satisfying the conditions of Case (I). Therefore there are two such extensions up to isomorphism. Once again to get the total number of non-equivalent simple wild parameters with image D_{2p} we need to count the number of automorphisms of $D_{2p} \leq \mathrm{PGL}_2(\mathbb{C})$ that do not come from conjugating by an element of $\mathrm{PGL}_2(\mathbb{C})$.

Lemma 2.22. *Let $\mathcal{D}$ be the group of automorphisms of $D_{2p} \leq \hat{G}$ that emerge as conjugation by an element of $\hat{G}$. Then $\mathrm{Aut}(D_{2p})/\mathcal{D}$ has order $\frac{p-1}{2}$.*

Proof. Let $D_{2p} = \langle \sigma, \tau \mid \sigma^{2p} = \tau^2 = 1, \tau\sigma\tau\sigma = 1\rangle$. As in Lemma 2.20, we have that $\mathrm{Aut}(D_{2p}) = C_{2p} \ltimes (C_{2p})^\times \simeq C_{2p} \ltimes C_{p-1}$. This time, conjugating by σ^p is the trivial automorphism, so the inner automorphism group of D_{2p} has order $2p$. However

$$\sigma^{1/2} := \begin{pmatrix} \sqrt{\zeta_p} & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{PGL}_2(\mathbb{C})$$

defines an automorphism of D_{2p} by acting by conjugation. Therefore $\mathcal{D}$ is generated by $\sigma^{1/2}$ and $\mathrm{Inn}(D_{2p})$ and has order $4p$, from which it follows that $\mathrm{Aut}(D_{2p})/\mathcal{D}$ has order $\frac{p-1}{2}$. $\square$

In conclusion, we have proved

Proposition 2.23.

1. *There is a unique simple wild parameter for* $\mathrm{SL}_2(\mathbb{Q}_2)$,
2. *There are $2 \cdot (p-1)$ simple wild parameters for* $\mathrm{SL}_2(\mathbb{Q}_p)$ *where p is any odd prime. There are $p-1$ with image isomorphic to D_{2p} and $p-1$ with image isomorphic to D_p.*

Later, we will match these to so-called simple supercuspidal representations.

3 Bruhat-Tits theory

In this section we introduce some elements of Bruhat-Tits theory, defining apartments and buildings associated to a p-adic reductive group G. The main concept needed from this section is that of parahoric subgroups. Nonetheless, we hope that the theory introduced provides some context for where these parahoric subgroups appear.

Once again, k is a finite extension of $\mathbb{Q}_p$ with normalized discrete valuation v_k, ring of ideals $\mathcal{O}_k$, and maximal ideal $\mathfrak{p}_k$.

3.1 Apartments

A reference for this section is [TBW79, §1.1].

Let G be simply connected, almost simple, and split over k. Fix a choice of maximal k-split torus T in G. Let N be the normalizer of T in G and T_0 the maximal compact subgroup of $T(k)$. Let Φ be the set of roots of T in G, with simple roots $\Delta = \{\alpha_1, \dots, \alpha_l\}$. Fixing a Chevalley basis $\{e_\alpha, h_{\alpha_i} \mid \alpha \in \Phi, 1 \le i \le l\}$ for the Lie algebra of G determines for each $\alpha \in \Phi$ an embedding

$$x_\alpha \colon k^+ \hookrightarrow G(k)$$

such that $dx_\alpha(1) = e_\alpha$. This map satisfies $tx_\alpha(c)t^{-1} = x_\alpha(\alpha(t)c)$ for $t \in T(k)$, $c \in k$. The image is denoted by U_α and is called the *root group* associated to α. In particular this is unipotent and normalized by $T(k)$.

Recall that the *Weyl group* with respect to T is $W = W(G,T)(k) = N(k)/T(k)$ and can be identified with the reflections $\langle s_\alpha \mid \alpha \in \Phi\rangle$. We have

$$\begin{array}{cc} W \hookrightarrow \mathrm{Aut}(X^*(T)) & W \hookrightarrow \mathrm{Aut}(X_*(T)) \\ (nT(k)\cdot\chi)(t) = \chi(n^{-1}tn) & (nT(k)\cdot\gamma)(x) = n^{-1}\gamma(x)n \end{array}$$

for $\chi \in X^*(T)$ and $\gamma \in X_*(T)$.

Consider the affine Euclidean space under the vector space $V = \mathbb{R} \otimes X_*(T)$. Each $\psi = \alpha + n$ where $\alpha \in \Phi$, $n \in \mathbb{Z}$ determines a non-zero affine linear functional on this affine space given by

$$x \mapsto \langle \alpha + n, x\rangle = \langle \alpha, x\rangle + n.$$

Here $\langle \alpha, x\rangle$ is defined by $\mathbb{R}$-linearly extending the pairing $X^*(T) \times X_*(T) \to \mathbb{Z}$.

Definition 3.1. The *apartment* of G associated to T is the affine Euclidean space $\mathcal{A}$ under the vector space V, equipped with a system $\Psi = \{\alpha + n \mid \alpha \in \Phi,\ n \in \mathbb{Z}\}$ of *affine roots* on $\mathcal{A}$, which are the affine linear functionals defined above.

We get an action of W on $\mathcal{A}$ by $\mathbb{R}$-linearly extending the action of W on $X_*(T)$. The choice of Chevalley basis determines an origin $O \in \mathcal{A}$. This is the unique point in $\mathcal{A}$ fixed by $s_\alpha = x_\alpha(-1)x_{-\alpha}(1)x_\alpha(-1) \in N(k)/T(k)$ for all $\alpha \in \Phi$. This allows us to view our apartment as a vector space.

For each affine root $\psi = \alpha + n$, we define the *affine root group* $U_\psi = x_\alpha(\mathfrak{p}_k^n) \subset G(k)$. Note that for $n \ge 0$, $U_{\alpha+n}$ gives a filtration of $U_\alpha(\mathcal{O}_k)$ corresponding to the usual filtration of $\mathcal{O}_k$. For $\psi = \alpha + n$, define the following hyperplane of $\mathcal{A}$

$$H_\psi = \{x \in \mathcal{A} \mid \langle x, \psi\rangle = 0\}.$$

An *alcove* is a connected component of $\{x \in \mathcal{A} \colon \langle \psi, x\rangle \ne 0\ \forall \psi \in \Psi\}$. We define $s_\psi \colon \mathcal{A} \to \mathcal{A}$ to be the orthogonal reflection about the hyperplane H_ψ. Explicitly

$$s_\psi(x) = x - (\langle \alpha, x\rangle - n)\,\alpha^\vee,$$

where $\alpha^\vee \in X_*(T)$ is the co-root satisfying $\langle \alpha, \alpha^\vee \rangle = 2$. The group generated by these reflections is the *affine Weyl group*

$$W(\Psi) = \langle s_\psi \colon \psi \in \Psi \rangle.$$

The group $W(\Psi)$ acts simply transitively on the set of alcoves (see [Bou02, Ch V, §3.2]).

The apartment has a natural action by $N(k)$. Firstly we show how $T(k)$ acts on $\mathcal{A}$. Consider a change of our Chevalley basis by $\operatorname{Ad} s$ for $s \in T(k)$. Then if $\psi = \alpha + n$ one has

$$\begin{array}{rcl} x_\alpha & \mapsto & x'_\alpha(y) = x_\alpha(\alpha(s) \cdot y), \\ U_\psi & \mapsto & U'_\psi = U_{\psi + v_k(\alpha(s))}. \end{array}$$

Observe that the terms of the filtration of $U_\alpha(\mathcal{O}_k)$ are unchanged, but the index has undergone a translation. We get a new unique point $y \in \mathcal{A}$ fixed by $x'_\alpha(-1)x'_{-\alpha}(1)x'_\alpha(-1)$ for all $\alpha \in \Phi$, and we let $s \in \operatorname{Aff}(\mathcal{A})$ be translation to the new origin; given by $s(x) = x - y$ for $x \in \mathcal{A}$. One can observe that $T(k)$ acts by translations on $\mathcal{A}$ such that, for $s \in T(k)$,

$$s^{-1} U_\alpha s = U_{\alpha \circ s}$$

where $\alpha \circ s$ is the affine root given by $\langle \alpha \circ s, x \rangle = \langle \alpha, s(x) \rangle$ for $x \in \mathcal{A}$. In particular, viewing $s \in \operatorname{Aff}(\mathcal{A})$, one has $s(x) = x + \nu(s)$ such that $\langle \alpha, \nu(s) \rangle = -v_k(\alpha(s))$ for all $\alpha \in \Phi$.

Example 3.2. Consider $k = \mathbb{Q}_p$ and $G = \mathrm{SL}_2(k)$, with Chevalley basis $\{e, f, h\}$ as in §2.2. Let T be the diagonal matrices in G. Then $\Phi = \{\alpha, -\alpha\}$, where $\alpha\left(\left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right)\right) = t^2, -\alpha\left(\left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right)\right) = t^{-2}$. One has $x_{\pm\alpha} \colon k^+ \to G(k)$ with

$$x_\alpha(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \quad x_{-\alpha}(t) = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}.$$

Then

$$s_{-\alpha} = s_\alpha = x_\alpha(-1)x_{-\alpha}(1)x_\alpha(-1) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and the unique point fixed by s_α is 0. Consider $s = \left(\begin{smallmatrix} p & 0 \\ 0 & p^{-1} \end{smallmatrix}\right)$. Then $\alpha(s) = p^2$ and $(-\alpha)(s) = p^{-2}$ and so changing our Chevalley basis by $\operatorname{Ad} s$ gives

$$s'_\alpha = x'_\alpha(-1)x'_{-\alpha}(1)x'_\alpha(-1) = \begin{pmatrix} 0 & -p^2 \\ p^{-2} & 0 \end{pmatrix}.$$

Our new origin is the unique point fixed by s'_α, which is $\alpha^\vee$, and so $s \in \operatorname{Aff}(\mathcal{A})$ translates by $-\alpha^\vee$, i.e. $s(x) = x - \alpha^\vee$ for $x \in \mathcal{A}$. Observe that $s^{-1}U_\alpha s = U_{\alpha - 2}$, and $\langle \alpha, x - \alpha^\vee \rangle = \langle \alpha, x \rangle - 2$ for $x \in \mathcal{A}$, so as expected we have $\alpha \circ s = \alpha - 2$.

Similarly, $N(k)$ acts on $\mathcal{A}$ via affine transformations such that $n^{-1}U_\alpha n = U_{\alpha \circ n}$ holds for all $s \in N(k)$ and $\alpha \in \Phi$. This action of $N(k)$ on $\mathcal{A}$ identifies $N(k)/T_0$ with $W(\Psi)$; if $n \in N(k)$ has image w in $W(\Psi)$, then $nU_\psi n^{-1} = U_{w\psi}$.

Example 3.3. Let $G = \mathrm{SL}_2(\mathbb{Q}_p)$, and $s = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Then $s^{-1}U_\alpha s = U_{-\alpha}$, $s^{-1}U_{-\alpha}s = U_\alpha$. Thus $s \in \operatorname{Aff}(\mathcal{A})$ acts as $s(x) = -x$. If $s = \left(\begin{smallmatrix} p & 0 \\ 0 & p^{-1} \end{smallmatrix}\right)$, then $s(x) = x - \alpha^\vee$. If $s = \left(\begin{smallmatrix} 0 & p \\ -p^{-1} & 0 \end{smallmatrix}\right)$, then $s(x) = -x - \alpha^\vee$.

3.2 Parahoric subgroups and buildings

Having defined introduced the apartment of G associated to a torus T, we can define certain subgroups of $G(k)$ known as parahoric subgroups and Iwahori subgroups. These are the analogues of parabolic subgroups and Borel subgroups respectively for complex Lie groups. See [Fin23, §3] for more details.

Definition 3.4. Let G, $\mathcal{A}$, Ψ, k be as before.

1. For $x \in \mathcal{A}$, define $G_x(k) = \langle T(\mathcal{O}_k),\ U_\psi \mid \psi \in \Psi, \langle \psi, x\rangle \geq 0\rangle \subset G(k)$. This is *the parahoric subgroup associated to* x.
2. Define $G_x^+(k) = \langle T(1+\mathfrak{p}_k),\ U_\psi \mid \psi \in \Psi, \langle \psi, x\rangle > 0\rangle \subset G(k)$. This is the pro-unipotent radical of G_x.
3. If $x \in \mathcal{A}$ is not contained in any hyperplane H_ψ of $\mathcal{A}$ for all affine roots ψ, then $G_x(k)$ is also called a *Iwahori subgroup*.

Parahoric subgroups are compact and open with respect to the topology on $G(k)$ induced by that of k. More explicitly,

$$G_x(k) = \left\langle T(\mathcal{O}_k),\ U_\alpha\left(\mathfrak{p}_k^{-\lfloor \langle \alpha, x\rangle \rfloor}\right) \mid \alpha \in \Phi \right\rangle, \quad G_x^+(k) = \left\langle T(1+\mathfrak{p}_k),\ U_\alpha\left(\mathfrak{p}_k^{1-\lceil \langle \alpha, x\rangle \rceil}\right) \mid \alpha \in \Phi \right\rangle.$$

Example 3.5. Let $G = \mathrm{SL}_2(\mathbb{Q}_p)$. The point $x = \frac{1}{4}\alpha^\vee$ has $\langle \alpha, \frac{1}{4}\alpha^\vee\rangle = \frac{1}{2}$ and $\langle -\alpha, \frac{1}{4}\alpha^\vee\rangle = -\frac{1}{2}$. Then

$$G_x = \langle T(\mathbb{Z}_p), U_\alpha(p\mathbb{Z}_p), U_{-\alpha}(\mathbb{Z}_p)\rangle = \begin{pmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p^\times \end{pmatrix}$$

and this is an Iwahori subgroup.

In the previous section, we could act on $\mathcal{A}$ by $N(k)$ since this normalized our choice of torus. The group $G(k)$ acts on a bigger object known as the the building associated to G. We use [BT72, §7] as a reference.

Definition 3.6. Let $\mathcal{A}$ be an apartment of G. Define the *building* $\mathcal{B}(G,k) = (G(k) \times \mathcal{A})/\sim$ where $(g, y) \sim (h, z)$ if $\exists n \in N(k)$ such that $z = n(y)$ and $g^{-1}hn \in G_y(k)$. Here $n(\cdot)$ denotes the action of n on $\mathcal{A}$.

Remark 3.7. If $n \in N(k)$, then we claim that $n \cdot G_y(k) \cdot n^{-1} = G_{n(y)}(k)$. Indeed $n \cdot T(\mathcal{O}_k) \cdot n^{-1} = T(\mathcal{O}_k)$ since $n \in N(k)$, and we have $n \cdot U_\psi \cdot n^{-1} = U_{\psi \circ n^{-1}}$ for $\psi \in \Psi$. Therefore $U_\psi \subset n \cdot G_y(k) \cdot n^{-1} \iff \langle \psi \circ n, y\rangle = \langle \psi, n(y)\rangle \geq 0 \iff U_\psi \subset G_{n(y)}(k)$. Using this property, one can verify that $\sim$ in Definition 3.6 defines an equivalence relation.

One has $\mathcal{A} \to \mathcal{B}(G,k)$ via $y \mapsto (1, y)$. This map is injective. Indeed if $(1, y) \sim (1, z)$ for $y, z \in \mathcal{A}$, then $\exists n \in G_y(k) \cap N(k)$ with $z = n(y)$. This implies that $z = y$ (see [BT72, §7.4] for details). This allows us to identify $\mathcal{A}$ with its image in $\mathcal{B}(G,k)$.

One can define an action of $G(k)$ on $\mathcal{B}(G,k)$ by letting $G(k)$ act on $G(k) \times \mathcal{A}$ by $g \cdot (h, y) = (gh, y)$, and verifying that this action passes to the quotient $\mathcal{B}(G,k)$. Since G is simply connected, we have the following interpretation of our parahoric groups:

Proposition 3.8 ([BT72, Proposition 7.4.4, Remark 7.1.10])**.** *Let* $x \in \mathcal{A} \subset \mathcal{B}(G,k)$. *Then* $\mathrm{Stab}_G(x) = G_x(k)$.

More generally, we define the following subgroups of $G(k)$.

Definition 3.9. For $x \in \mathcal{A}$ and $r \in \mathbb{R}_{\geq 0}$, define

$$G_{x,r}(k) = \left\langle T(1+\mathfrak{p}_k^{\lceil r \rceil}),\ U_\alpha(\mathfrak{p}_k^{-\lfloor \langle \alpha, x\rangle - r\rfloor}) \mid \alpha \in \Phi \right\rangle, \quad G_{x,r+}(k) = \cup_{s>r} G_{x,s}(k).$$

Observe $G_x(k) = G_{x,0}(k)$, $G_x^+(k) = G_{x,0}^+(k)$. For each $x \in \mathcal{A}$ we obtain a *Moy-Prasad filtration* of $G(k)$ given by $\{G_{x,r}(k) \mid r \in \mathbb{R}_{\geq 0}\}$. If $r \leq s$ then $G_{x,s}(k)$ is a normal subgroup of $G_{x,r}(k)$. In particular, $G_{x,r}(k) \triangleleft G_x(k)$ for all $r \geq 0$.

One can also define subgroups $G_{x,r}(k) \subset G(k)$ for any $x \in \mathcal{B}(G,k)$ and $r \in \mathbb{R}_{\geq 0}$. To do so, find an apartment $\mathcal{A} \subset \mathcal{B}(G,k)$ containing x. If this is the apartment of a split torus, we can use the definition above. Else, the definition is slightly different.

Remark 3.10 (Depth of a representation)**.** In [MP96, Theorem 3.5] Moy and Prasad define the *depth* of an irreducible admissible complex representation (π, V) of G. This is defined in terms of the subgroups $G_{x,r}(k)$. The depth $\rho(\pi) \in \mathbb{Q}_{\geq 0}$ satisfies the property that there exists $x \in \mathcal{B}(G, k)$ such that the space of invariants $V^{G_{x,\rho(\pi)+}(k)}$ is non-zero and it is the smallest rational number with this property.

4 Simple supercuspidal representations

This section covers the construction of simple supercuspidal representations, and calculates them for SL_2 over unramified extensions of $\mathbb{Q}_p$.

As the name suggests, simple supercuspidal representations are supercuspidal. While one does not need much knowledge of the representation theory of p-adic groups to follow the construction of simple supercuspidal representations, we will recall a few facts here. Sources on this topic include [Fin23, §2.4] and [Car79].

Let k be a finite extension of $\mathbb{Q}_p$, and consider a connected, reductive group G over k. Within the category $\mathrm{Smo}_{G(k)}$ of smooth complex representations of $G(k)$, there are *discrete series representations*. These are irreducible unitary representations such that their matrix coefficients are square-integrable with respect to some nonzero invariant measure μ on $G(k)$. A discrete series representation embeds into the regular representation of $G(k)$ on $L^2(G(k))$; the complex vector space of square-integrable functions on $G(k)$ with respect to μ.

A certain class of discrete series representations are the *supercuspidal representations*. These are discrete series representations whose matrix coefficients have compact support modulo the center $Z(k)$ of $G(k)$. These embed into the regular representation of $G(k)$ on $C_c^\infty(G(k))$; the space of functions on $G(k)$ that are locally constant and compactly supported.

Supercuspidal representations are of particular interest because they are the "building blocks" of the representation theory of p-adic groups in the following sense.

Theorem 4.1 ([Fin23, Fact 2.4.8])**.** *Let (π, V) be an irreducible smooth representation of $G(k)$. Then there exists a parabolic subgroup $P \subset G$ with Levi subgroup M and a supercuspidal representation (σ, W) of $M(k)$ such that π is a subrepresentation of $\mathrm{Ind}_{P(k)}^{G(k)} \sigma$.*

We also recall the notion of a central character of a representation. Let (π, V) be a discrete series representation. There is a character $\omega\colon Z(k) \to \mathbb{C}^\times$ such that $\pi(z)v = \omega(z)v$ for all $v \in V$ and $z \in Z(k)$. This character ω is called the *central character* of (π, V).

4.1 Construction of simple supercuspidal representations

In [GR10, §9.2, §9.3], Gross and Reeder give the construction for simple supercuspidal representations, which we detail here. These are obtained by inducing characters from a certain compact subgroup. In this section we assume that G is simply connected, almost simple, and split over k.

Let T be a k-split maximal torus of G with associated apartment $\mathcal{A}$ as defined in §3.1. Let T_0 be the maximal compact subgroup of $T(k)$. Fix an alcove C in $\mathcal{A}$ and a corresponding set of positive affine roots $\Psi^+ = \{\psi \in \Psi \colon \langle \psi, x\rangle \geq 0 \ \forall x \in C\}$. Then there is a unique subset $\Pi \subset \Psi^+$ such that any element of Ψ^+ is of the form $\sum_{\psi \in \Pi} n_\psi \cdot \psi$ with $n_\psi \geq 0$. We call elements of Π *simple affine roots*. There is a unique point $x_0 \in C$ on which all simple affine roots take the same value; this is the barycenter of C. This common value is $1/h$, where h is the Coxeter number of G.

As described in §3.2 $G_{x_0}(k)$ is an Iwahori subgroup, with pro-unipotent radical $G_{x_0}^+(k) = G_{x_0,1/h}$. We have

$$G_{x_0}(k)/G_{x_0,1/h}(k) \simeq T(q) = \{t \in T_0 \mid t^q = t\},$$

where q is the size of the residue field of k. The quotient $G_{x_0,1/h}(k)/G_{x_0,2/h}(k)$ can be described as a sum of quotients of root groups as follows.

Proposition 4.2 ([GR10, Lemma 9.2]). *One has the following isomorphism as T_0-modules*

$$G_{x_0,1/h}(k)/G_{x_0,2/h}(k) \simeq \bigoplus_{\psi\in\Pi} U_\psi/U_{\psi+1} \simeq \bigoplus_{\psi\in\Pi} \mathbb{F}_q.$$

Let Z denote the center of G. We construct our supercuspidal representations by inducing certain characters from the subgroup $Z(k)G_{x_0,1/h}(k)$. We have $Z(k)G_{x_0,1/h}(k) = Z(q) \times G_{x_0,1/h}(k)$ where $Z(q) = Z \cap T(q)$.

Definition 4.3 (Affine generic characters). An affine generic character is a character $\chi\colon Z(k)G_{x_0,1/h}(k) \to \mathbb{C}^\times$ such that

1. χ is trivial when restricted to $G_{x_0,2/h}(k)$,
2. χ is non-trivial when restricted to U_ψ for all $\psi \in \Pi$.

Let l be the rank of G. Then $|\Pi| = l+1$. It follows that there are $|Z(q)| \cdot (q-1)^{l+1}$ affine generic characters. As shown in the proof of [GR10, Proposition 9.4], the group $T(q)/Z(q)$ acts freely on the set of affine generic characters.

The subgroup $H(k) = Z(k)G_{x_0,1/h}(k)$ is compact and open in $G(k)$. For an affine generic character χ, we define the compactly induced representations

$$\pi_\chi = \text{c-Ind}_{H(k)}^{G(k)} \chi.$$

This consists of the space of functions $f\colon G(k) \to \mathbb{C}$ such that

1. $f(hg) = \chi(h)f(g)$ for all $h \in H(k)$, $g \in G(k)$,
2. The support of f is compact mod $H(k)$, i.e. the support of f consists of finitely many left cosets of $H(k)$ in $G(k)$.

Then $G(k)$ acts on a function f by $(g \cdot f)(h) = f(hg)$. It turns out that these representations are supercuspidal.

Proposition 4.4 ([GR10, Proposition 9.3]). *Let $\pi_\chi = \text{c-Ind}_{H(k)}^{G(k)}\chi$, where χ is an affine generic character. Then*

1. *π is an irreducible supercuspidal representation,*
2. *Let χ' be another affine generic character, and $\pi_{\chi'} = \text{c-Ind}_{H(k)}^{G(k)}\chi'$. Then π_χ and $\pi_{\chi'}$ are equivalent if and only if χ and χ are conjugate by an element of $T(q)$.*

We call the representation π_χ a *simple supercuspidal representation.* Since $T(q)/Z(q)$ acts freely on the affine generic characters, and $|T(q)| = (q-1)^l$ one therefore sees that there are $|Z(q)|^2(q-1)$ simple supercuspidal representations up to equivalence.

Remark 4.5. Gross and Reeder called these "simple" supercuspidal representations due to their relatively straightforward construction. Later Reeder and Yu ([RY14]) generalized this construction and defined *epipelagic representations*, which are certain supercuspidal representations of small positive depth.

The central character of a simple supercuspidal representation is easy to describe.

Proposition 4.6. *Let $\pi_\chi = \text{c-Ind}_{H(k)}^{G(k)}\chi$ be a simple supercuspidal representation. Then the central character $\omega\colon Z(k) \to \mathbb{C}^\times$ is given by $\chi|_{Z(k)}$.*

Proof. Consider an element $f\colon G(k) \to \mathbb{C}$ of $\text{c-Ind}_{H(k)}^{G(k)}\chi$. Let $z \in Z(k)$. Then $(z \cdot f)(h) = f(hz) = f(zh) = \chi(z)f(h)$ since $z \in Z(k) \subset H(k)$. Thus the action by z sends f to $\chi(z)f$ and so $\omega = \chi|_{Z(k)}$. $\square$

4.2 Calculating simple supercuspidals for SL_2

In this section we compute the simple supercuspidal representations of SL_2 over any unramified extension of $\mathbb{Q}_p$. Let $\tilde{F}$ denote a finite unramified extension of $\mathbb{Q}_p$ of degree n, with ring of integers $\mathcal{O}$, uniformizer ϖ, and unique maximal ideal $\mathfrak{P}$. Let $q = p^n$ be the order of the residue field $\mathcal{O}/\mathfrak{P}$.

Consider the maximal $\tilde{F}$-split torus T consisting of diagonal matrices. Let $\{\alpha, -\alpha\}$ denote the roots as in Example 3.2. The maximal compact subgroup of T is $T_0 = T(\mathcal{O}^\times)$. One has $X_*(T) \otimes \mathbb{R} = \{r\alpha^\vee \mid r \in \mathbb{R}\}$. We choose $C = (0, \frac{1}{2}\alpha^\vee)$ as our fundamental alcove. The corresponding simple affine roots are $\Pi = \{\alpha, \alpha_0 = 1 - \alpha\}$. Observe that for $x_0 = \frac{1}{4}\alpha^\vee \in C$, one has $\langle \alpha, x_0 \rangle = \langle \alpha_0, x_0 \rangle = 1/2$. Thus we consider the following subgroups of $G(\tilde{F})$.

$$G_{x_0,1/h}(\tilde{F}) = \begin{pmatrix} 1+\mathfrak{P} & \mathcal{O} \\ \varpi\mathcal{O} & 1+\mathfrak{P} \end{pmatrix}, \quad G_{x_0,2/h}(\tilde{F}) = \begin{pmatrix} 1+\mathfrak{P} & \varpi\mathcal{O} \\ \varpi^2\mathcal{O} & 1+\mathfrak{P} \end{pmatrix}.$$

Then $V = G_{x_0,1/h}(\tilde{F})/G_{x_0,2/h}(\tilde{F}) \simeq \mathbb{F}_q \times \mathbb{F}_q$ via the map

$$\begin{pmatrix} a & b \\ \varpi c & d \end{pmatrix} \mod G_{x_0,2/h}(\tilde{F}) \;\mapsto\; (b \mod \mathfrak{P},\; c \mod \mathfrak{P}).$$

There are q^2 distinct characters of V. We claim they are all of the form

$$\begin{array}{ccccccc} V & \xrightarrow{f} & \mathbb{F}_q & \xrightarrow{\mathrm{Tr}} & \mathbb{F}_p & \to & \mathbb{C}^\times \\ (b,c) & \mapsto & f(b,c) & \mapsto & \mathrm{Tr}(f(b,c)) & \mapsto & \zeta_p^{\mathrm{Tr}(f(b,c))}, \end{array}$$

where f is a linear functional on V, Tr is the additive trace map $\mathbb{F}_q \to \mathbb{F}_p$ that sends $x \in \mathbb{F}_q$ to $x + x^p + \cdots x^{p^{n-1}}$, and ζ_p is a primitive p-th root of unity. There are q^2 linear functionals on V, given by $f_{(x,y)}(b,c) = xb + yc$ where $x, y \in \mathbb{F}_q$. We claim that the resulting characters of V are distinct. Indeed, if $\zeta_p^{\mathrm{Tr}(xb+yc)} = \zeta_p^{\mathrm{Tr}(x'b+y'c)}$ for all $b,\, c \in \mathbb{F}_q$, then one has $\mathrm{Tr}(xb) - \mathrm{Tr}(x'b) = 0 \in \mathbb{F}_p$ for all $b \in \mathbb{F}_q$, that is that $\sum_{i=0}^{n-1} t^{p^i}(x^{p^i} - (x')^{p^i}) \in \mathbb{F}_q[t]$ has q roots. But this is of degree p^{n-1} in t, hence must be identically zero so that $x = x'$. A similar argument shows that $y = y'$.

We write $\chi_{(x,y)}$ for the character of V corresponding to the linear functional $f_{(x,y)}$. Firstly, assume that p is odd. Then the centre of $\mathrm{SL}_2(\tilde{F})$ is $Z(\tilde{F}) = \{\pm I_2\}$. Let $\chi_{(x,y),\pm}$ denote the two possible liftings of the character $\chi_{(x,y)}$ to $H(\tilde{F}) = Z(\tilde{F})G_{x_0,2/h}$, where $\chi_{(x,y),+}(-I_2) = 1$ and $\chi_{(x,y),-}(-I_2) = -1$. For $\chi_{(x,y),\pm}$ to be an affine generic character, it must be non-trivial when restricted to U_α and U_{α_0}, which is equivalent to requiring that $xy \neq 0 \in \mathbb{F}_q$. Then the representations

$$\pi_{(x,y),\pm} = \text{c-Ind}_{H(\tilde{F})}^{\mathrm{SL}_2(\tilde{F})} \chi_{(x,y),\pm}$$

are supercuspidal.

To count the number of simple supercuspidal representations up to equivalence, we count the orbits of $T(q)$ on the set of affine generic characters. Let $\tilde{t}$ denote the matrix in $T(q)$ with diagonal entries $t,\, t^{-1} \in \mu_{q-1}$. Then

$$\chi^{\tilde{t}}_{(x,y),\pm}\left(\begin{pmatrix} a & b \\ \varpi c & d \end{pmatrix}\right) = \chi_{(x,y),\pm}\left(\begin{pmatrix} a & t^2 b \\ t^{-2}\varpi c & d \end{pmatrix}\right).$$

Thus unless $t^2 = 1$, i.e. $\tilde{t} \in Z(\tilde{F})$, the twist by $\tilde{t}$ is a distinct affine generic character. It follows that $T(q)/Z(q)$ acts freely on the affine generic characters of $H(\tilde{F})$. One can take as orbit representatives

$$\{\chi_{s_y,+},\; \chi_{s_y,-},\; \chi_{n_y,+}, \chi_{n_y,-} \mid y \in \mathbb{F}_q^\times\}$$

where $s_y = (1, y)$, $n_y = (z, y)$ for z a fixed non-square in $\mathbb{F}_q^\times$. Thus there are $4(q-1)$ equivalence classes of these simple supercuspidal representations when p is odd, with representatives

$$\{\pi_{(1,y),+},\; \pi_{(1,y),-},\; \pi_{(z,y),+},\; \pi_{(z,y),-} \mid y \in \mathbb{F}_q^\times\}.$$

Now suppose $p = 2$. Then $Z(q) = \{\mathrm{I}_2\}$ and so $H(\tilde{F}) = G_{x_0,1/h}(\tilde{F})$. Then $T(q)$ acts transitively on the affine generic characters, and so there are $q-1$ simple supercuspidal representations

$$\pi_{(1,y)} = \text{c-Ind}_{G_{x_0,1/h}(\tilde{F})}^{G(\tilde{F})} \chi_{(1,y)}, \quad y \in \mathbb{F}_q^\times$$

of $G(\tilde{F})$ up to equivalence. We record this results of this section for the case of $\tilde{F} = \mathbb{Q}_p$.

Proposition 4.7. *Let $G = \mathrm{SL}_2$.*

1. *There is a unique simple supercuspidal representation of $G(\mathbb{Q}_2)$ up to equivalence,*
2. *For an odd prime p, there are $4(p-1)$ simple supercuspidal representations of $G(\mathbb{Q}_p)$ up to equivalence.*

5 Local Langlands correspondence

We will only focus on the parts of the local Langlands correspondence that are relevant for us. Let G be a split semisimple group over a non-archimedean local field k. First we define an enhancement of a Langlands parameter for G.

Definition 5.1 (Enhancement of a discrete Langlands parameter)**.** Given a discrete Langlands parameter $\varphi\colon W_k \times \mathrm{SL}_2(\mathbb{C}) \to \hat{G}$, an *enhancement* of φ is an irreducible representation of $A_\varphi = \mathrm{Cent}_{\hat{G}}(\mathrm{Im}\,\varphi)$.

Let $\Pi^2(G/k)$ be the set of equivalence classes of irreducible discrete series representations of $G(k)$. Let $\mathcal{L}(G/k)$ be the set of $\hat{G}$-conjugacy classes of pairs (φ, ρ) where $\varphi\colon W_k \times \mathrm{SL}_2(\mathbb{C}) \to \hat{G}$ is a discrete parameter and $\rho \in \mathrm{Irr}(A_\varphi)$ is an enhancement of φ. The Local Langlands correspondence states the following.

Conjecture 5.2 (Local Langlands Correspondence (LLC), [GR10, Conjecture 7.1])**.** *Let G be a connected split semisimple group over k. There exists a bijection*

$$\Pi^2(G/k) \to \mathcal{L}(G/k), \quad \pi \mapsto (\varphi_\pi, \rho_\pi)$$

with the following properties.

1. *(Central character) If π, $\pi' \in \Pi^2(G/k)$ have $\varphi_\pi = \varphi_{\pi'}$ then π and π' have the same central character.*
2. *(Supercuspidal packets) For a given discrete parameter φ, the following are equivalent*
   - *(a) All $\pi \in \Pi^2(G/k)$ with $\varphi_\pi = \varphi$ are supercuspidal,*
   - *(b) If $\varphi_\pi = \varphi$ and $\rho_\pi = 1$ then π is supercuspidal,*
   - *(c) $\varphi(\mathrm{SL}_2(\mathbb{C})) = 1$.*

This conjectural correspondence partitions irreducible discrete series representations into what are known as L-packets.

Definition 5.3 (L-packet)**.** Let G, k be as above, and let $\varphi\colon W_k \times \mathrm{SL}_2(\mathbb{C})$ be a discrete parameter. Then the set

$$\Pi_\varphi(G/k) = \{\pi \in \Pi^2(G/k) \colon \varphi_\pi = \varphi\}$$

is the *L-packet* associated to φ.

Another part of the LLC is the formal degree conjecture. We state the version reformulated by Gross and Reeder in [GR10]. This describes the formal degree of a discrete series representation π in terms of quantities associated to (φ_π, ρ_π), including the adjoint gamma factor of φ_π. The degree of π is a numerical invariant associated to π, one can think of it as a replacement of the notion of the dimension of a finite-dimensional representation. Unfortunately, we cannot explore this any further in this report, but its consideration was essential for the conjectures of Gross and Reeder in the following section.

Conjecture 5.4 (Formal degree conjecture, [GR10, Conjecture 7.1])**.** *The formal degree of $\pi \in \Pi^2(G/k)$, with respect to the Euler-Poincaré measure μ_G is given by*

$$(-1)^{r(G)} \deg_{\mu_G}(\pi) = \frac{\dim \rho_\pi}{|A_{\varphi_\pi}/\hat{Z}|} \cdot \frac{|\gamma(\varphi_\pi)|}{|\gamma(\varphi_0)|}.$$

where φ_0 is the principal parameter as in Example 2.6 and $r(G)$ is the rank of G over k.

5.1 LLC for simple supercuspidals and simple wild parameters

Let k be a finite extension of $\mathbb{Q}_p$, with residue field of size q. The formal degree conjecture imposes strict conditions on the Langlands parameters that correspond to simple supercuspidal representations. Assuming the formal degree conjecture, as well as some base-change relations for parameters, Gross and Reeder showed the following:

Conjecture 5.5 ([GR10, Proposition 9.4])**.** *Let G be a split simply connected group over k. Assume that the simple supercuspidal representation π of $G(k)$ corresponds to the pair (φ_π, ρ_π) as in Conjecture 5.2. Then φ_π is a simple wild parameter.*

Moreover if the characteristic p of k does not divide the order of the Weyl group of G, then φ_π satisfies the properties of Proposition 2.14 and A_{φ_π} is abelian with order equal to that of $Z(q)$. In addition $|Z(k)| = |Z(q)|$.

Gross and Reeder also conjecture how the simple supercuspidal representations should be partitioned into L-packets. In the case that G is connected, let $G_{\mathrm{ad}} = G/Z_G$ be the *adjoint group* of G. Given $t \in G_{\mathrm{ad}}(k)$ and a representation π of $G(k)$ let π^t be the representation of $G(k)$ given by

$$(\pi^t)(g) = \pi(t^{-1}gt), \quad g \in G(k).$$

Then letting $(t \cdot \pi) = \pi^t$ yields an action of $G_{\mathrm{ad}}(k)$ on the representations of $G(k)$.

Conjecture 5.6 ([GR10, §9.5 Remark])**.** *Let G be a split simply connected group over k. The simple supercuspidal representations of $G(k)$ are partitioned into $|Z(q)| \cdot (q-1)$ distinct L-packets, each of cardinality $|Z(q)|$ and consisting of a single $G_{\mathrm{ad}}(k)$-orbit.*

Under some assumptions on the residue characteristic of k, in [Kal13] Kaletha constructs a $G_{\mathrm{ad}}(k)$-orbit of simple supercuspidal representations from a simple wild parameter. In addition he shows that this $G_{\mathrm{ad}}(k)$-orbit of representations satisfies the expected properties of an L-packet.

Theorem 5.7 ([Kal13, §4, §5])**.** *Let G be a split simply connected group over k and assume that the characteristic p of k does not divide the order of the Weyl group of G. Let $\varphi\colon W_k \times \mathrm{SL}_2(\mathbb{C}) \to \hat{G}$ be a simple wild parameter. Then there is an explicit way to construct from φ a $G_{\mathrm{ad}}(k)$-conjugacy class of simple supercuspidal representations of size $|Z(k)|$.*

Moreover, there is a bijection between the $G_{\mathrm{ad}}(k)$-orbit of simple supercuspidal representations obtained and the characters of A_φ.

5.2 Showing the correspondence for $\mathrm{SL}_2(\mathbb{Q}_p)$

Let us first consider the case of $\mathrm{SL}_2(\mathbb{Q}_2)$. By Proposition 2.23, there is a unique simple wild parameter φ up to equivalence with image isomorphic to S_4. This has $A_\varphi = \{1\}$ and so there is only one enhancement of φ given by the trivial representation. On the other hand, by Proposition 4.7 we proved that there is a unique simple supercuspidal representation π of $\mathrm{SL}_2(\mathbb{Q}_2)$ up to equivalence. Therefore

$$\pi \leftrightarrow (\varphi, \mathrm{triv})$$

by the local Langlands correspondence and Conjecture 5.5 .

Now we consider $\mathrm{SL}_2(\mathbb{Q}_p)$ for p odd. Then by Proposition 2.23 there are $2(p-1)$ simple wild parameters up to equivalence, half of which have image isomorphic to D_p, and the other half with image isomorphic to D_{2p}. By Proposition 4.7, there are $4(p-1)$ simple supercuspidal representations of $\mathrm{SL}_2(\mathbb{Q}_p)$ up to equivalence. For each simple wild parameter φ, we have that $A_\varphi \simeq C_2$, and so there are two possible enhancements of φ. Thus on the level of counting, the $4(p-1)$ simple super cuspidal representations pair up into $2(p-1)$ L-packets of size two corresponding to our simple wild parameters, according to the local Langlands conjecture and Conjecture 5.5. The following proposition details the L-packets.

Proposition 5.8. *Consider the correspondence determined by Kaletha as in Theorem 5.7. Then the L-packets of the simple supercuspidal representations for $\mathrm{SL}_2(\mathbb{Q}_p)$ are of the form*

$$\{\pi_{(1,y),+},\ \pi_{(z,yz^{-1}),+}\}, \quad \{\pi_{(1,y),-},\ \pi_{(z,yz^{-1}),-}\}, \quad y \in \mathbb{F}_p^\times,$$

where $z \in \mathbb{F}_p^\times$ is a fixed non-square.

Proof. The L-packets are of size $2 = |Z(k)|$. Consider the element $t = \begin{pmatrix} \sqrt{z} & 0 \\ 0 & \sqrt{z}^{-1} \end{pmatrix} \in \mathrm{SL}_2(\overline{\mathbb{Q}}_p)$. Here we are viewing $z \in \mathbb{Z}_p$ as a Teichmüller lift from $\mathbb{F}_p$. Then the image of t in $G_{\mathrm{ad}}(\overline{\mathbb{Q}}_p)$ belongs to $G_{\mathrm{ad}}(\mathbb{Q}_p)$. Indeed, suppose $\sigma \in G_{\mathbb{Q}_p}$ acts non-trivially on $\sqrt{z}$, i.e. $\sigma(\sqrt{z}) = \sqrt{z}^{-1}$. Then $t^{-1}\sigma(t) = -\mathrm{I}_2 \in Z(\mathbb{Q}_p)$, so that $\sigma(t) \equiv t \mod Z(\mathbb{Q}_p)$.

We claim that $(\text{c-Ind}_{H(k)}^{G(k)}\chi)^t \simeq \text{c-Ind}_{H(k)}^{G(k)}\chi^t$ as representations of $G(k)$ for χ an affine generic character of $H(k)$. Firstly, one can compute that conjugation by t preserves $H(k)$, therefore χ^t is also a character of $H(k)$. The isomorphism is given by sending $f \in (\text{c-Ind}_{H(k)}^{G(k)}\chi)^t$ to $f^t : G(k) \to \mathbb{C}$, where $f^t(g) = f(t^{-1}gt)$. Then $f^t \in \text{c-Ind}_{H(k)}^{G(k)}$, since if $h \in H(k)$ and $g \in G(k)$, one has

$$f^t(hg) = f(t^{-1}hgt) = \chi(t^{-1}ht)f(t^{-1}gt) = \chi^t(h)f^t(h).$$

Moreover, the action of $G(k)$ is compatible; if g, $x \in G(k)$ then

$$(g \cdot f^t)(x) = f^t(xg) = f(t^{-1}xgt) = f(t^{-1}xtt^{-1}gt) = (g \cdot f)(t^{-1}xt) = (g \cdot f)^t(x).$$

The inverse map is then given by $f \mapsto f^{t^{-1}}$ for $f \in \text{c-Ind}_{H(k)}^{G(k)}\chi^t$. We have $\chi^t_{(1,y),+} = \chi_{(z,yz^{-1}),+}$ and so $\pi_{(1,y),+}$ and $\pi_{(z,yz^{-1}),+}$ lie in the same $G_{\mathrm{ad}}(\mathbb{Q}_p)$ orbit. Since the orbits are of size 2 it follows that they are as claimed, with the case of non-trivial character shown similarly. $\square$

5.2.1 Correspondence over $\mathbb{Q}_{p^2}$

To understand how our L-packets match to simple wild parameters, we consider the correspondence over a quadratic unramified extension of $\mathbb{Q}_p$. As detailed in Theorem 5.7, Kaletha constructs an explicit correspondence between simple wild parameters and $G_{\mathrm{ad}}(k)$-conjugacy classes of simple supercuspidal representations. Moreover, he proves that the correspondence he constructs has a natural compatibility with unramified extensions, which we now describe.

Let k be a non-archimedean local field of characteristic zero, and $\tilde{F}$ a finite unramified extension of k of degree n. Let G be a split simply connected group over k, and such that the residue characteristic p of k does not divide the order of the Weyl group of G, as in Theorem 5.7.

Proposition 5.9 ([Kal13, Proposition 6.1])**.** *Let $\varphi\colon W_k \to \hat{G}$ be a simple wild parameter for $G(k)$, and denote by $\tilde{\varphi}$ its restriction to $W_{\tilde{F}}$. Let $\pi =$ c-Indχ be a simple supercuspidal representation of $G(k)$ contained in the L-packet of φ.*

Let $\tilde{\pi} =$ c-Ind$\tilde{\chi}$ be a representation of $G(\tilde{F})$, where $\tilde{\chi}$ is the affine generic character obtained by composing χ with the norm map for the action of $\mathrm{Gal}(\tilde{F}/k)$ *on $Z(\tilde{F})G(\tilde{F})_{x,1/h}/G(\tilde{F})_{x,2/h}$. Then $\tilde{\pi}$ is contained in the L-packet of $\tilde{\varphi}$.*

We now apply this to $G = \mathrm{SL}_2$ with $k = \mathbb{Q}_p$ and $\tilde{F} = \mathbb{Q}_{p^2}$. Firstly, we consider the restriction of our Langlands parameters.

Proposition 5.10. *Let $\varphi\colon W_k \times \mathrm{SL}_2(\mathbb{C}) \to \hat{G}$ be a simple wild parameter such that $F = (\overline{k})^{\ker \varphi|_{W_k}}$ has* $\mathrm{Gal}(F/k) \simeq D_p$. *Then there is a unique simple wild parameter φ' such that $\tilde{\varphi} = \tilde{\varphi}'$. In addition one has $F(\zeta_{p^2-1}) = (\overline{k})^{\ker \varphi'|_{W_k}}$.*

Proof. Since $\tilde{\varphi}$ is discrete, there exists a finite extension $K/\mathbb{Q}_{p^2}$ such that $K = (\overline{k})^{\ker \varphi|_{W_{\mathbb{Q}_{p^2}}}}$. Then $\ker \phi \subset \ker \phi|_{W_{\mathbb{Q}_{p^2}}}$ so that $F \subset K$. Writing $W_k = \langle \mathrm{Fr} \rangle \ltimes I_{\mathbb{Q}_p}$, we have $W_{\mathbb{Q}_{p^2}} = \langle \mathrm{Fr}^2 \rangle \times I_k$ and so $\varphi(W_k) = \varphi(I_k) = \varphi(W_{\mathbb{Q}_{p^2}})$ since φ is totally ramified. Thus $\mathrm{Gal}(K/\mathbb{Q}_{p^2}) \simeq \varphi(W_{\mathbb{Q}_{p^2}}) = D_p$. Therefore we have the following diagram of subfields, noting $[K\colon F] = 2$ since $[K\colon \mathbb{Q}_p] = 4p$.

$$\begin{tikzcd} & K \arrow[dl, dash, "2"'] \arrow[dr, dash, "2p"] & \\ F \arrow[dr, dash, "2p"'] & & \mathbb{Q}_{p^2} \arrow[dl, dash, "2"] \\ & \mathbb{Q}_p & \end{tikzcd}$$

By multiplicity of ramification degrees, K/F is unramified, hence $K = F(\zeta_{p^2-1})$, noting that the size of the residue field of F is p.

In §2.2.2, we computed that there were $(p-1)/2$ simple wild parameters φ' such that $(\overline{k})^{\ker \varphi'|_{W_k}} = F(\zeta_{p^2-1})$. Since $F(\zeta_{p^2-1})/\mathbb{Q}_{p^2}$ is a finite extension, one has

$$\ker \varphi'|_{W_{\mathbb{Q}_{p^2}}} = \ker \varphi' \cap W_{\mathbb{Q}_{p^2}} = \mathrm{Gal}(\overline{\mathbb{Q}}_p/F(\zeta_{p^2-1})) \cap W_{\mathbb{Q}_{p^2}} = W(\overline{\mathbb{Q}}_p/F(\zeta_{p^2-1}))$$

by part 2 of Proposition 1.17. Then $\varphi'(W_{\mathbb{Q}_{p^2}}) \simeq W_{\mathbb{Q}_{p^2}}/W(\overline{\mathbb{Q}}_p/F(\zeta_{p^2-1})) \simeq \mathrm{Gal}(F(\zeta_{p^2-1})/\mathbb{Q}_{p^2})$ by part 3 of Proposition 1.17. Therefore $\tilde{\varphi}$ and $\tilde{\varphi}'$ factor through the same Galois extension. Then there is a unique φ' up to equivalence such that $\tilde{\varphi} = \tilde{\varphi}'$. ☐

This shows us that the Langlands parameters with image isomorphic D_p pair up with those with image isomorphic to D_{2p} when we restrict to $W_{\mathbb{Q}_{p^2}}$. Since the center of SL_2 over $\mathbb{Q}_{p^2}$ has size 2, the L-packets for simple supercuspidal representations of $\mathrm{SL}_2(\mathbb{Q}_{p^2})$ have size 2. Therefore some simple supercuspidal representations of $\mathrm{SL}_2(\mathbb{Q}_p)$ must become equivalent under the above construction which gives simple supercuspidal representations of $\mathrm{SL}_2(\mathbb{Q}_{p^2})$.

Let $\chi_{(x,y),\pm}\colon Z(\mathbb{Q}_p)G_{x,1/2}(\mathbb{Q}_p) \to \mathbb{C}^\times$ be an affine generic character for $\mathrm{SL}_2(\mathbb{Q}_p)$, with $(x,y) \in \mathbb{F}_p^{\times 2}$. Let N denote the norm map

$$\begin{array}{rccc} N\colon & Z(\mathbb{Q}_{p^2})G_{x,1/h}(\mathbb{Q}_{p^2})/G_{x,2/h}(\mathbb{Q}_{p^2}) & \to & Z(\mathbb{Q}_p)G_{x,1/h}(\mathbb{Q}_p)/G_{x,2/h}(\mathbb{Q}_p) \\ & g & \mapsto & g\,\mathrm{Fr}(g). \end{array}$$

Note that this sends

$$\begin{pmatrix} a & b \\ \varpi c & d \end{pmatrix} \mapsto \begin{pmatrix} * & ab^p + bd^p \\ \varpi(a^p c + dc^p) & * \end{pmatrix} \equiv \begin{pmatrix} * & b^p + b \\ \varpi(c + c^p) & * \end{pmatrix} \mod G_{x,2/h}(\mathbb{Q}_p),$$

and $N(Z(\mathbb{Q}_{p^2})) = \{\mathrm{I}_2\}$. It follows that $\chi_{(x,y),\pm} \circ N$ is the affine generic character $\chi_{(x,y),+}$ on $Z(\mathbb{Q}_{p^2})G_{x,1/h}(\mathbb{Q}_{p^2})$, viewing $(x,y) \in \mathbb{F}_{p^2}^{\times 2}$.

Moreover, consider an L-packet of representations $\{\text{c-Ind}_{H(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}\chi_{(1,y),\pm},\ \text{c-Ind}_{H(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}\chi_{(z,zy^{-1}),\pm}\}$ of $\mathrm{SL}_2(\mathbb{Q}_p)$, where z is a non-square in $\mathbb{F}_p^\times$. Then $\text{c-Ind}_{H(\tilde{F})}^{G(\tilde{F})}\left(\chi_{(1,y),\pm} \circ N\right)$ and $\text{c-Ind}_{H(\tilde{F})}^{G(\tilde{F})}\left(\chi_{(z,yz^{-1}),\pm} \circ N\right)$ are equivalent as representations of $\mathrm{SL}_2(\mathbb{Q}_{p^2})$ since z has a square root when viewed in $\mathbb{F}_{p^2}^\times$, and so the characters are conjugate by an element of $T(q)$.

Therefore our process of obtaining a simple supercuspidal for $\mathrm{SL}_2(\mathbb{Q}_{p^2})$ from one for $\mathrm{SL}_2(\mathbb{Q}_p)$ is a $4-1$ map, sending the $\mathrm{SL}_2(\mathbb{Q}_p)$-representations of the two L-packets

$$\{\pi_{(1,y),+},\ \pi_{(z,yz^{-1}),+}\} \cup \{\pi_{(1,y),-},\ \pi_{(z,yz^{-1}),-}\}$$

to the simple supercuspidal representation $\pi_{(1,y),+}$ of $\mathrm{SL}_2(\mathbb{Q}_{p^2})$. Let φ be the Langlands parameter corresponding to the first L-packet, and φ' the parameter corresponding to the second. By Proposition 5.9, the restriction of these parameters to $W_{\mathbb{Q}_{p^2}}$ is equal. Therefore by Proposition 5.10 one of these parameters has image isomorphic to D_p, and the other D_{2p}.

Remark 5.11. One would like to additionally identify which parameter has image D_p or D_{2p}. Unfortunately we did not have the time to do this properly. However from further analysis of the process in [Kal13, §4], it appears that the L-packets with trivial central characters should correspond to parameters with image D_p, and the L-packets with non-trivial central characters correspond to the parameters with image D_{2p}.

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