Manchester talk
[Joint work with V. Dokchitser, H. Green, A. Morgan.]

§1. Background + results

A/K abelian variety over global field K.

BSD conjecture : $\mathrm{rk}\, A/K = \mathrm{ord}_{s=1} L(A/K, s)$.

$A(K) \simeq \mathbb{Z}^{\mathrm{rk} A/K} \oplus A(K)_{\mathrm{tors}}$.

+

Functional eqn conj : $L^*(A/K, 2-s) = w(A/K) \cdot L^*(A/K, s)$

$\hookrightarrow \in \{\pm 1\}$ global root num.

$\shortparallel$

Parity conjecture : $w(A/K) = (-1)^{\mathrm{rk}\, A/K}$

(upshot: no L-functions in statement)

Let $\mathrm{Sel}_{p^\infty}(A/K) = \varinjlim_n \mathrm{Sel}_{p^n}(A/K)$.

$= \ker(H^1(G_K, A[p^n]) \to \prod_v H^1(G_{K_v}, A(\bar{K}_v^{sep})))$

and $\mathcal{X}_p(A/K) = \mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{Sel}_{p^\infty}(A/K), \mathbb{Q}_p/\mathbb{Z}_p) \otimes \mathbb{Q}_p$

$(0 \to A(K) \otimes \mathbb{Q}_p/\mathbb{Z}_p \to \mathrm{Sel}_{p^\infty}(A/K) \to Ш(A/K)[p^\infty] \to 0)$.

Then $\dim_{\mathbb{Q}_p} \mathcal{X}_p(A/K) = \mathrm{rk}\, A/K + \#$ copies of $\mathbb{Q}_p/\mathbb{Z}_p$ in $Ш(A/K)[p^\infty]$.

p-parity conjecture : $w(A/K) = (-1)^{\underbrace{\dim_{\mathbb{Q}_p}(\mathcal{X}_p(A/K))}_{\text{"}p^\infty\text{-Selmer rank"}}}$

Known cases :

- [Dokchitser - Dokchitser, 2010] $A = E$ elliptic curves, $K = \mathbb{Q}$, all p ($p=2$ (Monsky))
- Any A, all p, K global fn. field (i.e. fin. ext. of $\mathbb{F}_p(t)$) [Trihan-Yasuda]

(also elliptic curves over totally real fields that are modular, ECs w/ K-rational p-isogeny)

Let τ be an Artin rep over K factoring through a finite Galois ext $\mathrm{Gal}(F/K)$.

BSD conj for twists : $\langle \tau, A(F)\otimes_{\mathbb{Z}} \mathbb{C} \rangle = \mathrm{ord}_{s=1} L(A,\tau,s)$

$\uparrow$ inner product of characters of $\mathbb{C}[G]$-reps $\qquad$ $\uparrow$ twisted L-fn

p-parity conj for twists : $w(A/K,\tau) = (-1)^{\langle \tau, \mathcal{X}_p(A/F)\rangle}$

<u>Ex</u>: $\tau = \mathbb{C}[G/H]$, $w(A/K,\tau) = w(A/F^H)$

$\langle \tau, \mathcal{X}_p(A/F)\rangle = \dim_{\mathbb{Q}_p} \mathcal{X}_p(A/F^H)$.

<u>Aim</u>: Use p-parity for twists to deduce p-parity.

Theorem [Trihan-Yasuda, 2014]: The p-parity conjecture for twists by orthogonal representations holds for all AVs / global function fields of char $\neq p$.

(ℓ-adic sheaf $\longleftrightarrow$ overconvergent F-isocrystal
ℓ-adic étale cohom. $\longleftrightarrow$ rigid cohom.)

Let A/K be a <u>semistable</u> AV over a <u>number field</u> K.
Fix $G = \mathrm{Gal}(F/K)$ and prime p. Dokchitser-Dokchitser's "Brauer relations + regulator constants" machinery proves p-parity for twists for a set of reps S_p of G (indep. of A)

Theorem (A. 2025): Let C/K be a hyperelliptic curve over a number field K. Fix a prime p and Galois ext $G = \mathrm{Gal}(F/K)$

Then $w(\mathrm{Jac}\,C/K,\tau) = (-1)^{\langle \tau, \mathcal{X}_p(\mathrm{Jac}\,C/F)\rangle}$

for all $\tau \in S_p$, where $\mathrm{Jac}\,C$ can have any (tame) reduction at all places $v \nmid 2p$.

<u>Rmk</u>: $\mathbb{1} \notin S_p$.

Ex : $G = D_{2p} \binom{\text{size}}{2p} S_p \supseteq \left\{ \mathbb{1} \oplus \det \overset{\varepsilon}{\sigma} \oplus \sigma \ : \ \sigma \text{ 2-dim. rep of } G \right\}$

$\Theta = \{1\} - 2 \cdot C_2 - C_p + 2G$

Theorem. [Application to p-parity for hyperelliptic curves] Let $C : y^2 = f(x)$ be a hyperelliptic curve over a number field K.

Let $F = K(\mathrm{Jac}\, C[2])$, $G = \mathrm{Gal}(F/K) = \mathrm{Gal}(f)$.

Assume that for all $p \le 2g+1$

- $\# Ш(\mathrm{Jac}\, C/K)[p^\infty] < \infty$
- $\mathrm{Jac}\, C/K$ has s.s. redn at all places above p.

If the parity conjecture is true $\forall$ $\mathrm{Jac}\, C/F^H$ when $H \le G$ is a 2-group, then it is true for $\mathrm{Jac}\, C/K$.

Corr: Enough to prove parity for $C : y^2 = f(x)$ when $\mathrm{Gal}(f)$ is a 2-group.

§2. Reducing to a local statement.

Let $J_C = \mathrm{Jac}\, C$ $\quad C/K$ hyp. curve. $G = \mathrm{Gal}(F/K)$

Aim: Prove $w(J_C, \tau) = (-1)^{\langle \tau, \mathcal{X}_p(J_C/F) \rangle}$

Know $w(J_C, \tau) = \prod_{v \text{ place of } K} w(J_C, \mathrm{Res}_{D_v} \tau)$

D_v: decomp. gp @ v.

If $\tau \in S_p$, $\exists$ $n_i \in \mathbb{Z}$, $H_i \le G$ s.t.

$$\langle \tau, \mathcal{X}_p(J_C/F) \rangle \equiv \sum_{v \text{ place of } K} \underbrace{\prod_i C_v(J_C/F^{H_i})^{n_i}}_{=: \lambda_{\tau,v}(J_C)} \mod 2$$

where $C_v(J_C/F^{H_i})$ is the product of Tamagawa numbers of $J_C/(F^{H_i})_w$ for all w above v in F^{H_i}.

<u>Strategy</u> : Prove <u>local statement</u> : ($\star$)

- C/K hyperelliptic curve over local field
- $G = Gal(\mathcal{F}/K)$.

$$\forall \tau \in S_p \subseteq Rep(G) \quad w(J_C, \tau) = h_v \cdot (-1)^{\lambda_{\tau,v}(J_C)}$$

Where $h_v \in \{\pm 1\}$ is an <u>error term</u> s.t. $\prod_v h_v = 1$.

<u>Rmk</u>. $\star$ easy when * $Gal(\mathcal{F}/K)$ cyclic

* J_C/K good redn.

<u>Prop</u> : Let K be a non-arch. local field of odd pos. char. C/K hyp. curve.

Then local statement holds $\star$ $\forall$ $Gal(\mathcal{F}/K)$ and p with $p \neq$ char K.

<u>Proof idea</u> . "Approximate" C/K by a curve $\bar{C}/K$ over a global function field K with

- $K_v \cong K$, $\bar{C}/K_v$ suff. close to C/K for some v
- Local statement holds for $\bar{C}/K_{v'}$ $\forall v' \neq v$.

Then use global twisted p-parity theorem to deduce local statement at v $\Rightarrow$ same for C/K.

§3. $\mathbb{Q}_p \rightarrow \mathbb{F}_p((t))$

Let $K|\mathbb{Q}_p$ be a finite ext. w/ p odd, residue field k, unif. π.

Let $\mathcal{F} = K(\zeta_m, \sqrt[m]{\pi})$ have residue field $\mathbb{F}$ represented by the roots of unity of order coprime to p.

Write $\tilde{\alpha}$ for the Teichmüller lift of $\alpha \in \mathbb{F}$ to $\mathcal{F}$.

Define $\phi : \mathcal{F} \to \mathbb{F}((\sqrt[m]{t}))$ by

$$\phi\left(\sum_{i \geq N} \tilde{a}_i \sqrt[m]{\pi}^{\,i}\right) = \sum_{i \geq N} a_i \sqrt[m]{t}^{\,i}$$

$\mathcal{F} = \mathbb{Q}_3$

This is not a homomorphism: $x = 1+3$ $y = 3$, $\mathbb{F} = \{0, \pm 1\}$

$$\phi(x+y) = \phi(1 - 3 + 9)$$
$$= 1 - t + t^2$$
$$\neq \phi(x) + \phi(y)$$

We have an isom. $\mathrm{Gal}(\mathcal{F}|K) \to \mathrm{Gal}(\mathbb{F}((\sqrt[m]{t}))/k((t)))$

$$\sigma : \begin{array}{l} \zeta_m \mapsto \zeta_m^{\,i} \\ \sqrt[m]{\pi} \mapsto \zeta_m^{\,j} \sqrt[m]{\pi} \end{array} \longmapsto \hat{\sigma} : \begin{array}{l} \overline{\zeta_m} \to \overline{\zeta_m}^{\,i} \\ \sqrt[m]{t} \mapsto \overline{\zeta_m}^{\,j} \sqrt[m]{t} \end{array}$$

for $\sigma \in \mathrm{Gal}(\mathcal{F}/K)$ and $x \in \mathcal{F}$, $\phi_{\mathcal{F}}(\sigma(x)) = \hat{\sigma}(\phi_{\mathcal{F}}(x))$

Theorem: Let $C : y^2 = c(x_1 - r_1) \cdots (x - r_n) / K$ with $r_i \in \mathcal{F}$.

Consider

$$\phi(C) \quad y^2 = \phi(c)(x - \phi(r_1)) \cdots (x - \phi(r_n)) / k((t))$$

Fix $\ell \neq p$ prime

Then

☆ holds for C/K, tame Galois exts $G = \mathrm{Gal}(L/K)$ and $\tau \in S_\ell \subseteq \mathrm{Rep}(G)$ $\iff$ ☆ holds for $\phi(C)/k((t))$, tame Galois exts $G = \mathrm{Gal}(L|k((t)))$ and $\tau \in S_\ell \subseteq \mathrm{Rep}(G)$

In other words,

root numbers + tamagawa numbers for C transfer under ϕ