Steenrod squares and the Steenrod algebra

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Abstract

In this project we study the Steenrod squares: cohomology operations introduced by Norman Steenrod. We introduce their construction, and describe two bases of the Steenrod algebra: the Serre-Cartan basis and the Milnor basis. We use the Steenrod squares to calculate the cohomology of certain Eilenberg-MacLane spaces, and also see their application in calculating stable homotopy groups of spheres.

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Introduction

The viewpoint of cohomology in terms of representable functors leads to the consideration of natural transformations between these functors. In this project, we consider the category of pointed CW complexes, with morphisms that are homotopy classes of pointed maps.

The reduced singular cohomology of CW complexes is representable, with representable functors the contravariant functors $[-, K(G,n)]$. This means that for any X a CW complex, G an abelian group and $n > 0$, there is a bijection between $H^n(X;G)$ and $[X, K(G,n)]$, where $K(G,n)$ is an Eilenberg-MacLane space and $[-,-]$ denotes the homotopy classes of pointed maps. Moreover, the statement of Brown's representability theorem tells us that all reduced cohomology theories on the category of CW complexes are representable.

Natural transformations between the functors $[-, K(G,m)]$ and $[-, K(H,n)]$ correspond to cohomology operations between cohomology groups $H^m(-;G)$ and $H^n(-;H)$. These are operations that commute with the morphisms induced by maps between spaces.

The cup-product gives the cohomology of a space a ring structure, and the cup-product square is an example of a cohomology operation. Like the cup-product, cohomology operations can be used to question the existence of maps between spaces, in particular when just looking at the cohomology groups is not enough.

We will focus on the Steenrod squares in this project. Norman Steenrod first introduced the Steenrod squares in 1947, these being cohomology operations between cohomology groups with $\mathbb{Z}_2$ coefficients. One can consider the existence of the Steenrod squares as a consequence of the fact that, while the cup-product is commutative for cohomology with $\mathbb{Z}_2$ coefficients, it is only commutative up to homotopy when considering cochains.

We will introduce the squares, describe their construction, and then go through the technicalities of determining the Serre-Cartan and Milnor bases of the Steenrod algebra.

Armed with this, we will turn to some applications. We will detail the calculations of Jean-Pierre Serre in calculating the $\mathbb{Z}_2$-cohomology of Eilenberg-MacLane spaces $K(\mathbb{Z}_2, n)$. Serre performed these calcuations by using his Serre spectral sequence for fibrations. The cohomology of these Eilenberg-MacLane spaces will be polynomial in certain compositions of the squares, and we will describe how this implies the Steenrod squares generate all cohomology operations between cohomology groups with $\mathbb{Z}_2$ coefficients.

We will then use our knowledge of the cohomology of $K(\mathbb{Z}_2, n)$ spaces to perform some classical calculations of the stable homotopy groups of spheres. We do this by successively constructing better approximations to the spheres, by which we mean spaces with better approximations to the $\mathbb{Z}_2$-cohomology of the spheres. This in turn will result in the spaces better approximating (a component of) the homotopy of the spheres also. This is similar to the method of "killing homotopy groups" performed by the school of Henri Cartan and Jean-Pierre Serre.

An important application of the Steenrod squares is their use in the Adams spectral sequence. We will not discuss this here, but explain briefly how it relates to the work in the project at the end.

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1 Cohomology operations

1.1 Brown representability theorem

Throughout, we will work in the category of pointed CW-complexes, with morphisms that are homotopy equivalence classes of continuous pointed maps between objects. We will denote this category by $\mathfrak{C}$.

Definition 1.1 (Eilenberg-MacLane space)**.** Let G be an abelian group, and $n \geq 0$. An Eilenberg-MacLane space $K(G,n)$ is a space such that $\pi_n(K(G,n)) = G$ and $\pi_j(K(G,n)) = 0$ for all other $j \neq n, 0$.

We can take $K(G,n)$ to be homotopy equivalent to a CW-complex. Then it can be shown that for fixed G and n, there is a unique $K(G,n)$ space up to homotopy equivalence. Thus, we are justified in talking about "the" Eilenberg-MacLane space $K(G,n)$ for fixed G and n.

For every CW complex X, there is a natural bijection between $H^n(X;G)$ and $[X, K(G,n)]$, the homotopy classes of pointed maps from X to an Eilenberg-MacLane space $K(G,n)$.

Theorem 1.2 ([1, Theorem 4.57])**.** *Let G be an abelian group, X a CW-complex, and $n > 0$. Then there is a natural bijection $T\colon [X, K(G,n)] \to H^n(X;G)$. T is given by $T([f]) = f^*(\iota)$ for a class $\iota \in H^n(K(G,n);G)$.*

Remark 1.3. A class $\iota \in H^n(K(G,n);G)$ with the property in the theorem is called a fundamental class. Since $K(G,n)$ is $(n-1)$-connected, the Hurewicz map $h\colon G = \pi_n(K(G,n)) \to H_n(K(G,n);\mathbb{Z})$ is an isomorphism. Then by the universal coefficient theorem, $H^n(K(G,n);G) \simeq \operatorname{Hom}(H_n(K(G,n);\mathbb{Z}),G) \oplus \operatorname{Ext}(H_{n-1}(K(G,n);\mathbb{Z}),G)$, with the second term vanishing. Therefore, we take $\iota \in H^n(K(G,n);G) \simeq \operatorname{Hom}(H_n(K(G,n);\mathbb{Z}),G)$ given by the inverse of the Hurewicz isomorphism.

The above theorem can be stated more generally for any reduced cohomology theory, which is the statement of Brown's representability theorem. To state this, we will introduce CW spectra:

Definition 1.4 (CW spectrum)**.** A CW spectrum is a sequence of pointed CW complexes $\{X_n\}_{n\geq 0} \subset \mathfrak{C}$ such that there are CW inclusions $\Sigma X_n \hookrightarrow X_{n+1}$ for all n.

We can give examples using the following two definitions:

Definition 1.5 (Reduced suspension)**.** Let $I = [0,1]$. The suspension SX of a space X is the quotient of $X \times I$ obtained by collapsing $X \times \{0\}$ to a point and $X \times \{1\}$ to another point. The reduced suspension ΣX of X is $\Sigma X = SX/(\{x_0\} \times I)$, that is, further collapsing the base point of X along the entire interval I.

Definition 1.6 (Loop space)**.** The loop space ΩX of a space X is the space of pointed maps $\operatorname{map}_*(S^1, X)$.

Remark 1.7.

1. Ω and Σ are covariant functors: $\Omega\colon \mathfrak{C} \to \mathfrak{C}$, $\Sigma\colon \mathfrak{C} \to \mathfrak{C}$,
2. We have that ΣX is homotopic to $X \wedge S^1$, the smash product of X with S^1,
3. For reduced cohomology there is the suspension isomorphism $\sigma\colon H^*(X;G) \to H^{*+1}(\Sigma X;G)$ given by reduced cross product with a generator of $H^1(S^1)$ (see [1, Section 3.2, pg223])

Theorem 1.8 (Suspension-loop adjunction)**.** *The loopspace functor Ω has Σ as a left adjoint. That is, for all CW complexes X, $K \in \mathfrak{C}$, we have that the homotopy equivalence classes of maps between ΣX and K are isomorphic to those of X and ΩK, that is, $[\Sigma X, K] \simeq [X, \Omega K]$.*

Remark 1.9. Note that $[\Sigma X, K]$ and $[X, \Omega K]$ have group structures (see [1, Section 4.3]).

Example 1.10 (Suspension spectrum)**.** Given $X \in \mathfrak{C}$ the suspension spectrum $\Sigma^\infty X$ of X has $X = \Sigma^n X$ (the n-fold reduced suspension of X), and $\Sigma X_n \to X_{n+1}$ is the identity map.

Example 1.11 (Ω-spectrum)**.** For a spectrum $\{E_n\}_{n\geq 0} \subset \mathfrak{C}$ with maps $\Sigma E_n \to E_{n+1}$ for all n, we can consider their adjoints $E_n \to \Omega E_{n+1}$. $\{E_n\}_{n\geq 0}$ is a Ω-spectrum if $E_n \xrightarrow{\sim} \Omega E_{n+1}$ is a weak homotopy equivalence for all n.

Theorem 1.12 (Brown's Representability theorem, [1, Theorem 4E.1])**.** *Any reduced cohomology theory on the category of basepointed CW-complexes is of the form $h^n(X) = [X, E_n]$ for an Ω-spectrum $\{E_n\}_{n\geq 0}$.*

Remark 1.13. The definition of a reduced cohomology theory can be found in [1, Section 4.E]. Loosely speaking a reduced cohomology theory is a sequence of functors h^n from $\mathfrak{C}$ to abelian groups for $n \in \mathbb{Z}$ with a suspension isomorphism $h^n(X) \simeq h^{n+1}(\Sigma X)$ for all $X \in \mathfrak{C}$. It also must satisfy axioms relating to homotopy invariance, exact sequences of pairs, and wedge sums of spaces.

Example 1.14. For Eilenberg-Maclane spaces, there is a homotopy equivalence $K(G, n) \simeq \Omega K(G, n+1)$. Using the adjunction $[\Sigma K(G, n), K(G, n+1)] \simeq [K(G, n), \Omega K(G, n+1)]$ we get adjoint maps $\Sigma K(G, n) \to K(G, n+1)$ and obtain the Ω-spectrum $\{E_n\}_{n\geq 0}$ with $E_n = K(G, n)$. This is the Ω spectrum that corresponds to reduced singular cohomology with G coefficients.

This view-point of cohomology in terms of homotopy classes of maps will help us understand cohomology operations.

1.2 Cohomology operations

The advantage of working with cohomology over homology is that the cup-product gives cohomology a ring structure. In the same vain, cohomology operations provide us with additional information.

Definition 1.15 (Cohomology operation)**.** For fixed $m, n \geq 0$ and G, H abelian groups, a cohomology operation Θ is a map between cohomology groups that can be applied to any $X \in \mathfrak{C}$, $\Theta_X\colon H^m(X; G) \to H^n(X; H)$. Θ satisfies the naturality property that for any map $f\colon X \to Y$ between spaces, the following diagram commutes:

$$\begin{tikzcd}
H^m(Y;G) \arrow[r, "\Theta_Y"] \arrow[d, "f^*"] & H^n(Y;H) \arrow[d, "f^*"] \\
H^m(X;G) \arrow[r, "\Theta_X"'] & H^n(X;H)
\end{tikzcd}$$

Note that for Θ_X we may drop the subscript X if it is clear that Θ is being applied to the cohomology of X from context.

Example 1.16. The cup-product square is a map $c\colon H^n(-; G) \to H^{2n}(-; G)$ sending $x \in H^n(X; G)$ to $x^2 \in H^{2n}(X; G)$. This is a cohomology operation since the cup-product is natural.

Another way of obtaining cohomology operations is through short exact sequences of coefficient groups:

Example 1.17 ([1, Section 3.E])**.** Let X be a CW-complex and $(C_n(X))_n$ its chain complex. We can apply $\mathrm{Hom}(C_n(X), -)$ to a short exact sequence of abelian groups

$$0 \to G \to H \to K \to 0$$

to obtain

$$0 \to C^n(X; G) \to C^n(X; H) \to C^n(X; K) \to 0$$

where we have exactness since $C_n(X)$ is free, so all Ext terms are zero.

There is an associated long exact sequence:

$$\cdots \to H^n(X; G) \to H^n(X; H) \to H^n(X; K) \to H^{n+1}(X; G) \to \cdots$$

whose connecting homomorphism $H^n(X; K) \to H^{n+1}(X; G)$ is called a Bockstein homomorphism.

Example 1.18. The short exact sequences $0 \to \mathbb{Z} \xrightarrow{\cdot p} \mathbb{Z} \to \mathbb{Z}_p \to 0$ and $0 \to \mathbb{Z}_p \xrightarrow{\cdot p} \mathbb{Z}_{p^2} \to \mathbb{Z}_p \to 0$ induce Bockstein homomorphisms $\beta_1 \colon H^n(X;\mathbb{Z}_p) \to H^{n+1}(X;\mathbb{Z})$ and $\beta_2 \colon H^n(X;\mathbb{Z}_p) \to H^{n+1}(X;\mathbb{Z}_p)$ respectively. We have $\rho\beta_1 = \beta_2$ where $\rho \colon H^*(-;\mathbb{Z}) \to H^*(-;\mathbb{Z}_p)$ is the reduction of coefficients mod p.

1.2.1 Yoneda lemma

We will prove the following description of cohomology operations:

Theorem 1.19. *[1, Proposition 4L.1] Fix m, $n \in \mathbb{Z}$, G, H groups. There is a bijection between the set of cohomology operations Θ between cohomology groups with $\Theta_X \colon H^m(X;G) \to H^n(X;H)$, and $H^n(K(G,m);H)$. The bijection is given by $\Theta \mapsto \Theta_{K(G,m)}(\iota)$, where ι is a fundamental class in $H^m(K(G,m);G)$.*

We prove this by stating and proving a more general result in category theory, the Yoneda lemma. Firstly, we introduce the concept of a natural transformation between contravariant functors.

Definition 1.20 (Natural transformation)**.** If F and G are contravariant functors between categories $\mathbf{C}$ and $\mathbf{D}$, a natural transformation η is a rule that assigns to each object $A \in \mathbf{C}$ a morphism $\eta_A \colon FA \to GA$ of $\mathbf{D}$ such that for all morphisms $g \colon X \to Y$ in $\mathbf{C}$, the following diagram commutes:

$$\begin{tikzcd} Y & & FY \arrow[r, "\eta_Y"] \arrow[d, "Fg"] & GY \arrow[d, "Gg"] \\ X \arrow[u, "g"] & & FX \arrow[r, "\eta_X"] & GX \end{tikzcd}$$

Example 1.21. Consider the category of pointed CW complexes $\mathfrak{C}$. Then, for $m \geq 0$ and G an abelian group, there are contravariant functors $F_{m,G} \colon \mathfrak{C}^{op} \to \mathbf{Set}$ given by $X \mapsto H^m(X;G)$. A natural transformation η between $F_{m,G}$ and $F_{n,H}$ is then such that for each $X, Y \in \mathfrak{C}$ and $g \colon X \to Y$ a map, we have

$$\begin{tikzcd} H^m(Y;G) \arrow[r, "\eta_Y"] \arrow[d, "g^*"] & H^n(Y;H) \arrow[d, "g^*"] \\ H^m(X;G) \arrow[r, "\eta_X"'] & H^n(X;H) \end{tikzcd}$$

so that η is a cohomology operation, and likewise cohomology operations $H^m(-;G) \to H^n(-;H)$ define natural transformations between functors $F_{m,G}$, $F_{n,H}$.

We give a contravariant argument of the Yoneda lemma, so as to apply it to cohomology operations directly. The covariant argument is analogous.

Theorem 1.22 (Yoneda lemma, [2, Theorem 8.1])**.** *Let $\mathbf{C}$ be a category, and X an object of $\mathbf{C}$. Let $h^X \colon \mathbf{C}^{\mathrm{op}} \to$ **Set** be the contravariant functor $h^X = \mathrm{Hom}(-,X)$. Then for any contravariant set-valued functor $F \colon \mathbf{C}^{\mathrm{op}} \to$ **Set**, we have a bijection between the natural transformations from h^X to F and $FX \in$ **Set**, that is,*

$$FX \simeq \mathrm{Nat}(h^X, F)$$

Proof. We will define maps $\tau \colon \mathrm{Nat}(h^X, F) \to FX$ and $\lambda \colon FX \to \mathrm{Nat}(h^X, F)$ and show that they are inverses of one another.

- Let $\eta \in \mathrm{Nat}(h^X, F)$, and let $1_X \in h^X X = \mathrm{Hom}(X,X)$ be the identity map. Then by naturality for any map $g \colon Z \to X$ and $Z \in \mathbf{C}$, we have

  $$\eta_Z(h^X g)(1_X) = (Fg)\eta_X(1_X).$$

  Then $(h^X g)(1_X) = 1_X g = g$ so $\eta_Z(g) = Fg(\eta_X(1_X))$. This encourages us to define $\tau \colon \mathrm{Nat}(h^X, F) \to FX$ by $\tau(\eta) = \eta_X(1_X)$.

- Let $x \in FX$. We define a natural transformation $\lambda(x) \in \mathrm{Nat}(h^X, F)$. For each $Z \in \mathbf{C}$ and $g\colon Z \to X \in h^X Z$, let $(\lambda(x))_Z\colon h^X Z \to FZ$ be given by

$$(\lambda(x))_Z(g) = (Fg)x.$$

$\lambda(x)$ is a natural trasnformation since for $Z, Y \in \mathbf{C}$ and $f\colon Z \to Y$, the following commutes:

$$\begin{tikzcd} Z \arrow[d, "f"'] & h^X Y \arrow[r, "(\lambda(x))_Y"] \arrow[d, "h^X f"] & FY \arrow[d, "Ff"] \\ Y & h^X Z \arrow[r, "(\lambda(x))_Z"] & FZ \end{tikzcd}$$

Indeed, for $g \in h^X Y$, $(\lambda(x))_Z h^X f(g) = (\lambda(x))_Z(gf) = (F(gf))x$, and $(Ff)(\lambda(x))_Y(g) = (Ff)(Fg)x = (F(gf))x$. We define $\lambda\colon FX \to \mathrm{Nat}(h^X, F)$ by $x \mapsto \lambda(x)$.

- Now we show τ and λ are inverses. For $\eta \in \mathrm{Nat}(h^X, F)$ and $g \in h^X Z$ for $Z \in \mathbf{C}$, we have $\lambda(\tau(\eta))(g) = \lambda(\eta_X(1_X))(g) = (Fg)(\eta_X(1_X)) = \eta_Z(g)$ by above. Thus $\lambda(\tau(\eta)) = \eta$. Conversely, for $x \in FX$, $\tau(\lambda(x)) = (\lambda(x))_X(1_X) = (F1_X)(x) = x$.

□

Corollary 1.23. *Let* $\mathbf{C}$ *be a category,* X, Y *objects in* $\mathbf{C}$. *Then there is a bijection*

$$\mathrm{Hom}(X, Y) \simeq \mathrm{Nat}\left(\mathrm{Hom}(-, X), \mathrm{Hom}(-, Y)\right)$$

Proof. Let $F = h^Y = \mathrm{Hom}(-, Y)$, and apply the Yoneda lemma. □

We are now in a position to prove Theorem 1.19.

Proof of Theorem 1.19. By CW-approximation, it suffices to prove the statement for the case of $X \in \mathfrak{C}$. Then we can identify $H^m(X; G)$ with $[X, K(G, m)]$ and likewise $H^n(X; H)$ with $[X, K(H, n)]$. By Corollary 1.23,

$$\begin{aligned} \mathrm{Hom}(K(G, m), K(H, n)) &\simeq \mathrm{Nat}\left(\mathrm{Hom}(X, K(G, m)), \mathrm{Hom}(X, K(H, n))\right) \\ &\simeq \mathrm{Nat}(H^m(X; G), H^n(X; H)), \end{aligned}$$

but the natural transformations between the cohomology groups are cohomology operations by example 1.21, and $\mathrm{Hom}(K(G, m), K(H, n))$ is $H^n(K(G, m); H)$.

Note that in the proof of Yoneda's lemma, the map τ sends a cohomology operation Θ to $\Theta_{K(G,m)}(\iota)$ for ι a fundamental class in $H^m(K(G, m); G)$. □

Corollary 1.24. *Cohomology operations cannot decrease degree.*

Proof. A cohomology operation $\Theta_Z\colon H^m(Z; G) \to H^n(Z; H)$ with $n < m$ corresponds to a cohomology class in $H^n(K(G, m); H)$. But $K(G, m)$ is $(m-1)$-connected. □

Example 1.25. The only cohomology operations that preserve degree are given by homomorphisms of the cohomology coefficient groups. Such cohomology operations are in bijection with $H^n(K(G, n); H)$. Then by the universal coefficient theorem and the Hurewicz isomorphism for $K(G, n)$ we have $H^n(K(G, n); H) \simeq \mathrm{Hom}(H_n(K(G, n); \mathbb{Z}); H) \simeq \mathrm{Hom}(G, H)$.

In example 1.16, the operation is the same for any value n. We have just seen that cohomology operations cannot lower the degree, and we are led to define the following:

Definition 1.26 (System of cohomology operations)**.** Let G be an abelian group. A system of cohomology operations Θ^i of degree i for $i \geq 0$ is a sequence of cohomology operations $\{\Theta^i(n)\}_{n \geq 0}$ where each $\Theta^i(n)$ is a cohomology operation $\Theta^i(n)\colon H^n(-; G) \to H^{n+i}(-; G)$.

We will typically call Θ^i a cohomology operation, where it is implicit that it is actually a system of cohomology operations, and write Θ^i rather than $\Theta^i(n)$ for each n.

1.2.2 Stable cohomology operations

For convenience, we'll consider cohomology operations between groups with the same coefficients from now on.

Definition 1.27 (Stable cohomology operation)**.** Let G be an abelian group, and Θ^i a system of cohomology operations that raise the degree by i. Θ^i is a stable cohomology operation if it commutes with suspension. That is, for any $X \in \mathfrak{C}$, we have the following commuting for all $n \geq 0$:

$$\begin{tikzcd} H^n(X;G) \arrow[r, "\Theta^i_X"] \arrow[d, "\sigma"] & H^{n+i}(X;G) \arrow[d, "\sigma"] \\ H^{n+1}(\Sigma X;G) \arrow[r, "\Theta^i_{\Sigma X}"] & H^{n+i+1}(\Sigma X;G) \end{tikzcd}$$

where $\sigma\colon H^n(X;G) \to H^{n+1}(\Sigma X;G)$ is the suspension isomorphism.

Example 1.28. The cup-product square in example 1.16 is not stable since suspending then squaring sends a class in degree n to $2(n+1)$, whereas squaring then suspending sends the class of degree n to $2n+1$.

Note that while a regular cohomology operation does not have to be a group homomorphism between cohomology groups, a stable one does. This is because stable cohomology operations are additive on suspensions of spaces.

To show this, we note that for $\alpha, \beta \in H^n(\Sigma X;G)$ corresponding to $f, g \in [X, K(G,n)]$, the sum $\alpha + \beta \in H^n(\Sigma X;G)$ corresponds to the map $\Sigma X \xrightarrow{p} \Sigma X \vee \Sigma X \xrightarrow{f \vee g} K(G,n)$ where p is the pinch map collapsing an equatorial copy of $X \subset \Sigma X$ to a point. Let $\alpha \vee \beta \in H^n(\Sigma X \vee \Sigma X;G)$ be the cohomology class corresponding to $f \vee g$. Then $p^*(\alpha \vee \beta) = \alpha + \beta$. We claim that there is an isomorphism $H^n(\Sigma X;G) \times H^n(\Sigma X;G) \xrightarrow{\simeq} H^n(\Sigma X \vee \Sigma X;G)$ sending $\alpha \times \beta$ to $\alpha \vee \beta$. For $\Theta = \Theta^i$ a stable cohomology operation, we then have the following commuting by naturality (we suppress the coefficient group G):

$$\begin{tikzcd} H^n(\Sigma X) \times H^n(\Sigma X) \arrow[r, "\simeq"] \arrow[d, "\Theta \times \Theta"] & H^n(\Sigma X \vee \Sigma X) \arrow[r, "p^*"] \arrow[d, "\Theta"] & H^n(\Sigma X) \arrow[d, "\Theta"] \\ H^{n+i}(\Sigma X) \times H^{n+1+i}(\Sigma X) \arrow[r, "\simeq"] & H^{n+i}(\Sigma X \vee \Sigma X) \arrow[r, "p^*"] & H^{n+i}(\Sigma X) \end{tikzcd}$$

Then $p^*(\Theta(\alpha) \vee \Theta(\beta)) = \Theta(p^*(\alpha \vee \beta))$, that is $\Theta(\alpha) + \Theta(\beta) = \Theta(\alpha + \beta)$. But since Θ is stable, we have that it must be additive on the cohomology of X also.

The cup-product square is a homomorphism for $\mathbb{Z}_2$ cohomology coefficients, and not for $\mathbb{Z}$ cohomology coefficients. We will see in Section 2 that we can extend the cup-product square to a stable cohomology operation over $\mathbb{Z}_2$, but this is impossible over $\mathbb{Z}$ by the above.

Remark 1.29. Let G be an abelian group. Consider the Eilenberg-MacLane Ω-spectrum $E = \{E_n\}_{n \geq 0}$ with $E_n = K(G,n)$ defined in example 1.14. Then $\Theta^i(n)$ corresponds to a map in $\Theta^i(n) \in [E_n, E_{n+i}]$ for all n.

We have that the set of stable cohomology operations that raise the degree by i are classified by the inverse limit of the groups $[E_n, E_{n+i}]$ where the maps $[E_{n+1}, E_{n+1+i}] \to [E_n, E_{n+i}]$ are given by:

1. Precomposing $E_{n+1} \to E_{n+1+i}$ with the structure map: $\Sigma E_n \to E_{n+1} \to E_{n+1+i}$,
2. Taking the adjoint of the composite $\Sigma E_n \to E_{n+1+i}$ yielding $E_n \to \Omega E_{n+1+i} \simeq E_{n+i}$.

Correspondingly, this is an inverse limit $\varprojlim_n \tilde{H}^{n+i}(K(\mathbb{Z}_2, n); \mathbb{Z}_2)$.

2 Steenrod squares and their construction

2.1 Introduction

We have seen how to characterize cohomology operations in terms of cohomology of Eilenberg-MacLane spaces. In this section we construct the Steenrod squares, particular cohomology operations which we will later show generate all stable cohomology operations for cohomology in $\mathbb{Z}_2$ coefficients.

There are analagous operations, Steenrod powers, for cohomology with $\mathbb{Z}_p$ coefficients for p a prime, but we will restrict our attention to the squares and $\mathbb{Z}_2$-cohomology here.

Given a space X, unless otherwise stated, we assume from now on that $H^*(X)$ refers to the cohomology of X with $\mathbb{Z}_2$ coefficients.

Definition 2.1 (Steenrod squares)**.** The Steenrod squares, operations between the cohomology of a space $\mathrm{Sq}^i\colon H^n(X) \to H^{n+i}(X)$, $i \geq 0$, are group homomorphisms satisfying the following properties:

1. Sq^i are natural: for $f\colon X \to Y$ one has $f^* \mathrm{Sq}^i = \mathrm{Sq}^i f^*$,
2. $\mathrm{Sq}^0 = \mathrm{Id}$, the identity,
3. $\mathrm{Sq}^i(\alpha) = \alpha^2$ if $i = |\alpha|$, and $\mathrm{Sq}^i(\alpha) = 0$ if $i > |\alpha|$,
4. $\mathrm{Sq}^i(\alpha \smile \beta) = \sum_j \mathrm{Sq}^j(\alpha) \smile \mathrm{Sq}^{i-j}(\beta)$ (the Cartan formula),
5. $\mathrm{Sq}^i(\sigma(\alpha)) = \sigma(\mathrm{Sq}^i(\alpha))$ where $\sigma\colon H^n(X) \to H^{n+1}(\Sigma X)$ is the suspension isomorphism,
6. Sq^1 is the $\mathbb{Z}_2$ Bockstein homomorphism β associated with the coefficient sequence $0 \to \mathbb{Z}_2 \to \mathbb{Z}_4 \to \mathbb{Z}_2 \to 0$ (from example 1.18),
7. Adém Relations: $\mathrm{Sq}^a \mathrm{Sq}^b = \sum_c \binom{b-c-1}{a-2c} \mathrm{Sq}^{a+b-c} \mathrm{Sq}^c$ for $a < 2b$ and binomial coefficients taken mod 2.

Remark 2.2. The Cartan formula shows that the total square defines a ring homomorphism on $H^*(X)$. The total square is defined as $\mathrm{Sq} = \mathrm{Sq}^0 + \mathrm{Sq}^1 + \cdots$ which acts on $H^*(X)$. This is well-defined since on any cohomology class, Sq has only finitely many non-zero terms. Then we have $\mathrm{Sq}(\alpha \smile \beta) = \mathrm{Sq}(\alpha) \smile \mathrm{Sq}(\beta)$.

Remark 2.3. The last three properties are consequences of the first four. One can also construct the squares on relative cohomology, so that it satisfies the first four properties. Our construction later will just focus on the squares defined on $H^n(X)$.

Lemma 2.4. *The Cartan formula also holds for the external cross-product:* $\mathrm{Sq}^i(x \times y) = \sum_j \mathrm{Sq}^j x \times \mathrm{Sq}^{i-j} y$ *for* $x \times y \in H^*(X \times Y)$.

Proof. Let $p_1\colon X \times Y \to X$ and $p_2\colon X \times Y \to Y$ be projections. Then $x \times y = (x \times 1) \smile (1 \times y)$ and so $\mathrm{Sq}^i(x \times y) = \sum_j \mathrm{Sq}^j(x \times 1) \smile \mathrm{Sq}^{i-j}(1 \times y)$. But $x \times 1 = p_1^*(x)$ and likewise for y, so that by naturality $\mathrm{Sq}^i(x \times y) = \sum_j p_1^*(\mathrm{Sq}^i(x)) \smile p_2^*(\mathrm{Sq}^{i-j}(y)) = \sum_j (\mathrm{Sq}^i x \times 1) \smile (1 \times \mathrm{Sq}^{i-j} y) = \sum_j \mathrm{Sq}^i x \times \mathrm{Sq}^{i-j} y$. ∎

Later we will use that the squares commute with coboundary maps for pairs:

Lemma 2.5 ([3, Lemma 1.2])**.** *Axioms* 1, 2, *and* 4 *imply that if* $\delta\colon H^q(A) \to H^{q+1}(X, A)$ *is the coboundary map, then* $\delta \mathrm{Sq}^i = \mathrm{Sq}^i \delta$.

The Adém relations involve binomial coefficients mod 2, the following result will help us in calculations with these:

Lemma 2.6 (Lucas' theorem)**.** *Let p be a prime, and let a and b have the p-adic expansions $a = \sum_{i=0}^{m} a_i p^i$, $b = \sum_{i=0}^{m} b_i p^i$ where $0 \leq a_i, b_i < p$. Then,*

$$\binom{b}{a} = \prod_{i=0}^{m} \binom{b_i}{a_i} \qquad (\mathit{mod}\ p)$$

Proof. In the polynomial ring $\mathbb{Z}_p[x]$, $(1+x)^p = 1+x^p$. Thus $(1+x)^b = \prod_i (1+x)^{b_i p^i} \equiv \prod_i (1+x^{p^i})^{b_i}$. Now $\binom{b}{a}$ is the coefficient of x^a in this expansion, as is seen from the first expression; but the inspection of the last expression shows that this coefficient is precisely $\prod \binom{b_i}{a_i}$. $\square$

Remark 2.7. For $a, b \in \mathbb{N}$ we just need to find the binary expressions of a, b to compute $\binom{a}{b}$ mod 2. We have $\binom{0}{0} = 1$, $\binom{0}{1} = 0$, $\binom{1}{0} = 1$, and $\binom{1}{1} = 1$.

Example 2.8. Consider $X = \mathbb{R}P^\infty$. This is a $K(\mathbb{Z}_2, 1)$, since it has S^∞, which is contractible, as a universal double cover. We have $H^*(X) \simeq \mathbb{Z}_2[\alpha]$, the polynomial ring with α a generator in $H^1(X)$. Then $\mathrm{Sq}(\alpha) = \alpha + \alpha^2$, and so by the Cartan formula, $\mathrm{Sq}(\alpha^k) = \alpha^k(1+\alpha)^k = \sum_i \binom{k}{i}\alpha^{k+i}$, hence by comparing degrees, $\mathrm{Sq}^i(\alpha^k) = \binom{k}{i}\alpha^{k+i}$.

In particular, letting $k = 2^l$ and using Lucas' theorem 2.6 we have,

$$\mathrm{Sq}^i(\alpha^{2^l}) = \begin{cases} \alpha^{2^l} & \text{for } i = 0, \\ \alpha^{2^{l+1}} & \text{for } i = 2^l, \\ 0 & \text{otherwise.} \end{cases}$$

Example 2.9. We can use Sq^2 to show that $\pi_{n+1}S^n \neq 0$ for $n \geq 2$, by showing that suspensions of the Hopf map $\eta\colon S^3 \to S^2$ never become null-homotopic. The Hopf map is the attaching map of a cell to $\mathbb{C}P^1 \simeq S^2$ to obtain $\mathbb{C}P^2$. Therefore, we equivalently show that suspensions of $\mathbb{C}P^2$ never become homotopic to a wedge of spheres.

We show $\Sigma\mathbb{C}P^2 \not\simeq S^3 \vee S^5$. Both spaces have the same integral cohomology, a copy of $\mathbb{Z}$ in degree 0, 3, 5, and zero elsewhere. We have $H^*(\mathbb{C}P^2) = \mathbb{Z}_2[a]/(a^3)$ with $|a| = 2$, and the cup-product square sends the degree 2 cohomology class a to the non-trivial a^2. Since the squares commute with suspension, $\mathrm{Sq}^2(\sigma(a)) = \sigma(a^2)$, which is non-zero, hence Sq^2 acts non-trivially on the cohomology of $\Sigma\mathbb{C}P^2$. By the Künneth formula the generator of $H^3(S^3 \vee S^5)$ is $b \otimes 1$ for $b \in H^3(S^3)$ and $1 \in H^0(S^5)$. Therefore Sq^2 acts trivially on this class, so that $\Sigma\mathbb{C}P^2 \not\simeq S^3 \vee S^5$.

The same argument shows that $\Sigma^k\mathbb{C}P^2$ is never a wedge of spheres.

2.2 Construction of squares

We follow the construction presented in [1, Section 4.L].

For $n > 0$, let K_n denote a $K(\mathbb{Z}_2, n)$ CW complex with $(n-1)$-skeleton a point and n-skeleton S^n.

Let X be a pointed CW complex. Let $x \in H^n(X)$ be represented (up to homotopy) by a map $f \in [X, K(\mathbb{Z}_2, n)]$. Then $x^2 \in H^{2n}(X)$ can be represented by a map $g \in [X, K(\mathbb{Z}_2, 2n)]$ such that g is given by the composition

$$g\colon X \xrightarrow{\Delta} X \times X \xrightarrow{f \times f} K_n \times K_n \to K_n \wedge K_n \xrightarrow{\theta_0} K_{2n}$$

(this comes from $x^2 = \Delta^*(x \times x)$). The multiplication map θ_0 can be considered as the representative of a cohomology class in $H^{2n}(K_n \wedge K_n) = \mathrm{Hom}(H_{2n}(K_n \wedge K_n); \mathbb{Z}_2) = \mathrm{Hom}(\mathbb{Z}_2 \otimes \mathbb{Z}_2, \mathbb{Z}_2)$, e.g. taking the multiplication homomorphism $\mathbb{Z}_2 \otimes \mathbb{Z}_2 \to \mathbb{Z}_2$.

We know that the cup-product square is commutative for cohomology in $\mathbb{Z}_2$ coefficients. This means that the maps g and g' are homotopic, where

$$g'\colon X \xrightarrow{\Delta} X \times X \xrightarrow{f \times f} K_n \times K_n \to K_n \wedge K_n \xrightarrow{T} K_n \wedge K_n \xrightarrow{\theta_0} K_{2n}$$

with T permuting the factors of $K_n \wedge K_n$. Thus θ_0 is symmetric up to homotopy, but in general not actually symmetric.

The map T generates a $\mathbb{Z}_2$ action on $K_n \wedge K_n$. The map θ_0 does not pass to the orbit space. We want to instead consider a space Y with a $\mathbb{Z}_2$ action and a map $\theta\colon Y \to K_{2n}$ that does pass to the orbit space of Y under the $\mathbb{Z}_2$ action. This will work if the action of $\mathbb{Z}_2$ on Y is free, and moreover to relate back to $K_n \wedge K_n$ we would like Y to be homotopy equivalent to $K_n \wedge K_n$.

The answer is to take what is known as a Borel construction, or homotopy quotient, of $K_n \wedge K_n$. The contractible space S^∞ has a free $\mathbb{Z}_2$ action given by sending a point to its antipode, with $S^\infty/\mathbb{Z}_2 \simeq \mathbb{R}P^\infty$. Therefore if we take the space $(K_n \wedge K_n) \times S^\infty$, the diagonal $\mathbb{Z}_2$ action acts freely on this space. We aim to find some map $\theta\colon (K_n \wedge K_n) \times S^\infty \to K_{2n}$, that will pass to the orbit space as $\theta'\colon (K_n \wedge K_n) \times_{\mathbb{Z}_2} S^\infty \to K_{2n}$.

If we do, we can now consider:

$$X \times S^\infty \xrightarrow{\Delta\times 1} (X\times X)\times S^\infty \xrightarrow{(f\times f)\times 1} (K_n\times K_n)\times S^\infty \to (K_n\wedge K_n)\times S^\infty \to (K_n\wedge K_n)\times_{\mathbb{Z}_2} S^\infty \xrightarrow{\theta'} K_{2n}$$

or, truncating things a little,

$$X \times S^\infty \xrightarrow{\Delta\times 1} (X \times X) \times S^\infty \to (K_n \wedge K_n) \times_{\mathbb{Z}_2} S^\infty \xrightarrow{\theta'} K_{2n}$$

This is almost the set-up we want. Let $k_0 \wedge k_0$ be the basepoint of $K_n \wedge K_n$. The section $S^\infty \to (K_n \wedge K_n) \times S^\infty \to S^\infty$ given by sending $s \mapsto (k_0 \wedge k_0, s)$ induces a section $\mathbb{R}P^\infty \to (K_n \wedge K_n) \times_{\mathbb{Z}_2} S^\infty \to \mathbb{R}P^\infty$, where the inclusion maps $\mathbb{R}P^\infty$ into $(k_0 \wedge k_0) \times \mathbb{R}P^\infty$. We quotient out by this section to obtain the space ΛK_n. Suppose that θ' passes to a map $\lambda\colon \Lambda K_n \to K_{2n}$. Now we get

$$X \times S^\infty \xrightarrow{\Delta\times 1} (X \times X) \times S^\infty \to \Lambda K_n \xrightarrow{\lambda} K_{2n}$$

But $\Delta(X \times X)$ is fixed by the $\mathbb{Z}_2$ action that permutes its factors. Thus $(\Delta(X) \times S^\infty)/\mathbb{Z}_2 = X \times \mathbb{R}P^\infty$, and we can factor out this $\mathbb{Z}_2$ action and obtain a map

$$X \times \mathbb{R}P^\infty \to \Lambda K_n \xrightarrow{\lambda} K_{2n}$$

Finally, because we quotiened out $(k_0 \wedge k_0) \times \mathbb{R}P^\infty$ to obtain ΛK_n, we can take the quotient map $X \times \mathbb{R}P^\infty \to X \wedge \mathbb{R}P^\infty$ to obtain

$$X \wedge \mathbb{R}P^\infty \to \Lambda K_n \xrightarrow{\lambda} K_{2n}$$

This map represents a cohomology class in $H^{2n}(X \wedge \mathbb{R}P^\infty)$. Then by the Künneth theorem for reduced cohomology, $H^{2n}(X \wedge \mathbb{R}P^\infty) \simeq \sum_{i=0}^{n} H^{n+i}(X) \otimes H^{n-i}(\mathbb{R}P^\infty)$. Recall that we started with a cohomology class $x \in H^n(X)$. We define $\mathrm{Sq}^i(x)$ to be the cohomology class in $H^{n+i}(X)$ in our summation.

Essentially, what we are doing is taking what is known as the equivariant cohomology of $\Delta(X\times X)\times S^\infty$. This means we are taking the cohomology of the homotopy quotient of $\Delta(X\times X)$ considered as a $\mathbb{Z}_2$-space, which is $H^*(X \times \mathbb{R}P^\infty)$. We go from $\mathbb{Z}_2$-equivariant maps $[X \times X, K_{2n}]$ to maps $[X \times \mathbb{R}P^\infty, K_{2n}]$. Then we just fix things so that the maps are all base-point preserving.

The central task is to construct the map $\theta\colon (K_n \wedge K_n) \times S^\infty \to K_{2n}$, and such a map that passes to the quotient spaces we want it to. We will do this by finding a homotopy and extending it, and so we will need to refer to the following extension lemma:

Lemma 2.10 (Extension lemma, [1], Lemma 4.7)**.** *Given a CW pair (X, A) and a map $f\colon A \to Y$ with Y path-connected, then f can be extended to a map $X \to Y$ if $\pi_{n-1}(Y) = 0$ for all n such that $X - A$ has cells of dimension n.*

Now we will do things more explicitly, and sometimes for convenience in a slightly different order than what was detailed above.

Step 1: Homotopy quotient.
Let (X, x_0) be a pointed CW complex. Define $T: X \wedge X \to X \wedge X$, $T((x_1, x_2)) = (x_2, x_1)$, which generates a $\mathbb{Z}_2$ action on $X \wedge X$.

We can consider a $\mathbb{Z}_2$ action sending the coordinates of S^∞ to their antipode such that $S^\infty/\mathbb{Z}_2 \simeq \mathbb{R}P^\infty$. This action is free.

We take the diagonal action of $\mathbb{Z}_2$ on the product space $S^\infty \times (X \wedge X)$. Let ΓX denote the orbit space $S^\infty \times_{\mathbb{Z}_2} (X \wedge X)$.

The projection $S^\infty \times (X \wedge X) \to S^\infty$ induces a projection $\pi: \Gamma X \to \mathbb{R}P^\infty$ with $\pi^{-1}(z) \simeq X \wedge X$ for all $z \in \mathbb{R}P^\infty$ since the action of $\mathbb{Z}_2$ on S^∞ is free. Thus we have that $\pi: \Gamma X \to \mathbb{R}P^\infty$ is a fiber bundle with fiber $X \wedge X$.

The $\mathbb{Z}_2$ action on $X \wedge X$ fixes the basepoint $x_0 := (x, x_0) \sim (x_0, x)$ so that the inclusion $S^\infty \times \{x_0\} \hookrightarrow S^\infty \times X \wedge X$ induces an inclusion of the quotient spaces: $\mathbb{R}P^\infty \hookrightarrow \Gamma X$. The composition $\mathbb{R}P^\infty \hookrightarrow \Gamma X \to \mathbb{R}P^\infty$ is the identity, so $\mathbb{R}P^\infty \subset \Gamma X$ is a section of the bundle. Let ΛX denote the quotient of ΓX, obtained by collapsing $\mathbb{R}P^\infty$ to a point. The fibers $X \wedge X$ are still embedded in the quotient ΓX since each fiber meets the section $\mathbb{R}P^\infty$ at a single point; the basepoint.

We claim that these constructions are still CW complexes. $\mathbb{R}P^\infty$ and S^∞ are given their usual cellular structures. T freely permutes the product cells of $S^\infty \times (X \wedge X)$, and T is cellular, so there is an induced quotient CW structure on ΓX. The section $\mathbb{R}P^\infty \subset \Gamma X$ is a subcomplex, so the quotient ΛX inherits a CW structure from ΓX.

We also define $\Gamma^1 X$, the orbit space of the same diagonal action of $\mathbb{Z}_2$ on $S^1 \times (X \wedge X)$. Then $\Gamma^1 X$ is a subcomplex of ΓX. Likewise we define $\Lambda^1 X$, which is obtained from $\Gamma_1 X$ by quotienting out the section $S^1/\mathbb{Z}_2 \simeq S^1$.

Note that for a pointed map $\alpha: X \to Y$, we can define $\alpha': S^\infty \times X \wedge X \to S^\infty \times Y \wedge Y$ with $\alpha'(s, x, y) = (s, \alpha(x), \alpha(y))$. Then α' passes to ΓX so that we can define $\Gamma\alpha: \Gamma X \to \Gamma Y$, and this passes further to ΛX, so that we obtain $\Lambda\alpha: \Lambda X \to \Lambda Y$ such that the following commutes:

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "\Lambda"] & Y \arrow[d, "\Lambda"] \\ \Lambda X \arrow[r, "\Lambda\alpha"] & \Lambda Y \end{tikzcd}$$

Similarly we can obtain $\Gamma_1\alpha$ and $\Lambda_1\alpha$.

Step 2: Constructing and extending a homotopy.
Let $\iota_n \in H^n(K_n)$ be the fundamental class such that the homotopy class $[K_n, K_n]$ representing ι_n contains Id_{K_n}.

We have a reduced cross product $H^*(K_n) \otimes H^*(K_n) \to H^*(K_n \wedge K_n)$, which is an isomorphism by Künneth (see [1, Theorem 3.21]). Therefore $K_n \wedge K_n$ is $(2n-1)$-connected. Then $\iota_n \otimes \iota_n \in H^n(K_n) \otimes H^n(K_n) \simeq H^{2n}(K_n \wedge K_n)$ can be represented by a map $u: K_n \wedge K_n \to K_{2n}$. Considering $T: K_n \wedge K_n \to K_n \wedge K_n$ that permutes the factors, we have:

Lemma 2.11. *$u: K_n \wedge K_n \to K_{2n}$ is homotopic to uT.*

Proof. The $2n$-skeleton of $K_n \wedge K_n$ is $S^n \wedge S^n = S^{2n}$. Consider the restriction $T|_{S^{2n}}$. T is cellular, so we can consider $(T|_{S^{2n}}) \circ (T|_{S^{2n}})$ which is the identity map $S^{2n} \to K_{2n}$. Since $\pi_{2n}(K_{2n}) = \mathbb{Z}_2$, it follows that $T|_{S^{2n}}$ is homotopic to the identity. Therefore u and uT are homotopic on this restriction. $S^{2n} \subset K_n \wedge K_n$ is a sub-complex, so $(K_n \wedge K_n, S^{2n})$ is a CW pair. There are no obstructions to extending our homotopy over all higher dimensional cells of $K_n \wedge K_n$ by the extension lemma 2.10, since $\pi_i(K_{2n}) = 0$ for $i > 2n$. □

The point of our construction of Λ is that we can extend this homotopy to a map between ΛK_n and K_{2n}. Indeed, the homotopy $u \simeq uT$ defines a map $H: I \times K_n \wedge K_n \to K_{2n}$ such that $H(0, x, y) = H(1, y, x)$. If we concatenate the homotopies H and HT, then we get a loop homotopy $H': S^1 \times K_n \wedge K_n \to K_{2n}$.

Considering $S^1 \simeq [0,1]/(0 \sim 1)$, and the antipodal action of $\mathbb{Z}_2$ on S^1 as a translation of $[0,1]$ by $1/2$, we get that $H'(s,x,y) = H'(-s,y,x)$. Thus we get a map $\tilde{H}\colon \Gamma^1 K_n \to K_{2n}$.

The homotopy is basepoint preserving, so the map $\Gamma^1 K_n \to K_{2n}$ passes down to a quotient map $\lambda_1\colon \Lambda^1 K_n \to K_{2n}$. Since K_n is obtained from S^n by attaching cells of dimension greater than n, ΛK_n is obtained from $\Lambda^1 K_n$ by attaching cells of dimension greater than $2n+1$. There are then no obstructions to extending λ_1 to a map $\lambda\colon \Lambda K_n \to K_{2n}$ by lemma 2.10, since $\pi_i(K_{2n}) = 0$ for $i > 2n$.

The map λ gives a cohomology class $\lambda^*(\iota_{2n}) \in H^{2n}(\Lambda K_n)$. If we consider the restriction of λ to $K_n \wedge K_n$, then this is homotopic to u, so that this restriction represents the cohomology class $\iota_n \otimes \iota_n \in H^{2n}(K_n \wedge K_n)$.

The following point is important in proving the properties of the squares:

Remark 2.12. The property that the restriction of λ to $K_n \wedge K_n$ is homotopic to u determines λ uniquely up to homotopy, since the map $H^{2n}(\Lambda K_n) \to H^{2n}(K_n \wedge K_n)$ is injective (the $2n$ skeleton of ΛK_n being contained in $K_n \wedge K_n$).

Step 3: Defining the squares.
We considered the space K_n since we have the *natural* isomorphism in thereom 1.2. Any $\omega \in H^n(X)$ corresponds to a map $\rho \in [X, K_n]$. Let us denote by $\lambda(\omega) \in H^{2n}(\Lambda X)$ the cohomology class represented by the composition $\Lambda X \xrightarrow{\Lambda\rho} \Lambda K_n \xrightarrow{\lambda} K_{2n}$. The restriction of this composition in each fiber $X \wedge X$ represents the cohomology class $\omega \otimes \omega \in H^{2n}(X \wedge X)$, since λ restricts to u in $K_n \wedge K_n$ and $u(\rho \wedge \rho)$ represents the cohomology class $(\rho\wedge\rho)^* u^*(\iota_{2n}) = (\rho\wedge\rho)^*(\iota_n \otimes \iota_n) = \rho^*(\iota_n) \otimes \rho^*(\iota_n) = \omega \otimes \omega$.

We want to find some way of relating the cohomology of X to that of ΛX. There is an inclusion $\mathbb{R}P^\infty \wedge X \hookrightarrow \Gamma X$ as the quotient of the diagonal embedding $S^\infty \wedge X \hookrightarrow S^\infty \wedge X \wedge X$, $(s,x) \mapsto (s,x,x)$. Composing with the quotient map $\Gamma X \to \Lambda X$, we get a map $\nabla\colon \mathbb{R}P^\infty \wedge X \to \Lambda X$ inducing $\nabla^*\colon H^*(\Lambda X) \to H^*(\mathbb{R}P^\infty \wedge X) \simeq H^*(\mathbb{R}P^\infty) \otimes H^*(X)$. That is, we have:

$$\begin{tikzcd} & X \arrow[r, "\rho"] \arrow[d, "\Lambda"] & K_n \arrow[d, "\Lambda"] & \\ \mathbb{R}P^\infty \wedge X \arrow[r, "\nabla"] & \Lambda X \arrow[r, "\Lambda\rho"] & \Lambda K_n \arrow[r, "\lambda"] & K_{2n} \end{tikzcd}$$

Then, for each $\omega \in H^n(X)$, the element $\nabla^*(\lambda(\omega)) \in H^{2n}(\mathbb{R}P^\infty \wedge X)$ may be written as

$$\nabla^*(\lambda(\omega)) = \sum_{i=0}^{n} \alpha^{n-i} \otimes \mathrm{Sq}^i(\omega), \tag{1}$$

where α is a generator of $H^1(\mathbb{R}P^\infty)$ so that $H^*(\mathbb{R}P^\infty) = \mathbb{Z}_2[\alpha]$, and the second factor in each summation term is defined to be $\mathrm{Sq}^i(\omega)$ with $|\mathrm{Sq}^i(\omega)| = n+i$.

Note that for $\omega \in H^n(X)$ and $\rho \in [X, K_n]$ such that $\rho^*(\iota_n) = \omega$, we have $\mathrm{Sq}^i(\omega) = \rho^*(\mathrm{Sq}^i(\iota_n))$. Since the bijection in theorem 1.2 is natural, we have naturality of the squares in general, so that they really are cohomology operations.

The other properties take more work to prove. We refer the reader to [1, Section 4.L].

2.3 Adém relations

We are going to prove the following:

Theorem 2.13 ([1, Theorem 4L.13])**.** *The Adém relations hold for the Steenrod squares.*

The Adém relations detail the behaviour of the squares under composition. When $a < 2b$ we have:

$$\mathrm{Sq}^a \mathrm{Sq}^b = \sum_c \binom{b-c-1}{a-2c} \mathrm{Sq}^{a+b-c} \mathrm{Sq}^c .$$

If $a \geq 2b$, then no Adém relation can be applied to $\mathrm{Sq}^a \mathrm{Sq}^b$.

In our construction of the Steenrod squares, we went from a cohomology class in $H^*(X)$ to one in $H^*(\mathbb{R}P^\infty \wedge X)$. Iterating the process yields a class in $H^*(\mathbb{R}P^\infty \wedge \mathbb{R}P^\infty \wedge X)$. Let α_1, α_2 be the classes of degree 1 such that $H^*(\mathbb{R}P^\infty) \simeq \mathbb{Z}_2[\alpha_1]$ for the first $\mathbb{R}P^\infty$ factor and $H^*(\mathbb{R}P^\infty) \simeq \mathbb{Z}_2[\alpha_2]$ for the second. Then $H^*(\mathbb{R}P^\infty \wedge \mathbb{R}P^\infty X) \simeq H^*(X)[\alpha_1, \alpha_2]$. We will prove that the iteration of the Steenrod squares yields cohomology classes that are symmetric polynomials in $H^*(X)[\alpha_1, \alpha_2]$. We derive the Adém relations by consequence.

We look at the neatest case where $X = K_n$, and consider the iteration of the squares on the cohomology class $\omega = \iota_n \in H^n(K_n)$. Then $\rho \in [K_n, K_n]$ with $\rho^*(\iota_n) = \omega$ can be taken to be Id_{K_n}. Now to compute the class in $H^{4n}(\mathbb{R}P^\infty \wedge \mathbb{R}P^\infty \wedge K_n)$, we can use example 2.8 and the Künneth theorem. Indeed for $\alpha \in H^1(\mathbb{R}P^\infty)$ the generator, $\mathrm{Sq}^i(\alpha^n) = \binom{n}{i}\alpha^{n+i}$. We'll denote by $\lambda(\lambda)$ the class in $H^{4n}(\Lambda(\Lambda X))$ represented by the map $\Lambda(\lambda \Lambda \mathrm{Id}_{k_n})$ Therefore using (1), the cohomology class in $H^{4n}(\mathbb{R}P^\infty \wedge \mathbb{R}P^\infty \wedge X)$ is given by

$$\begin{aligned}
(\mathrm{Id} \wedge \nabla)^*(\nabla^*(\lambda(\lambda))) &= (\mathrm{Id} \times \nabla)^*(\textstyle\sum_i \alpha_1^{2n-i} \otimes \mathrm{Sq}^i(\lambda)) \\
&= \textstyle\sum_i \alpha_1^{2n-i} \otimes \nabla^*(\mathrm{Sq}^i \lambda^*(\iota_n)) \\
&= \textstyle\sum_i \alpha_1^{2n-i} \otimes \mathrm{Sq}^i(\nabla^*(\lambda^*(\iota_n))) \\
&= \textstyle\sum_i \alpha_1^{2n-i} \otimes \mathrm{Sq}^i(\sum_j \alpha_2^{n-j} \otimes \mathrm{Sq}^j(\iota_n)) \\
&= \textstyle\sum_{i,j} \alpha_1^{2n-i} \otimes \mathrm{Sq}^i(\alpha_2^{n-j} \otimes \mathrm{Sq}^j(\iota_n)) \\
&= \textstyle\sum_{i,j} \alpha_1^{2n-i} \otimes \sum_k \mathrm{Sq}^k(\alpha_2^{n-j}) \otimes \mathrm{Sq}^{i-k} \mathrm{Sq}^j(\iota_n) \\
&= \textstyle\sum_{i,j} \alpha_1^{2n-i} \otimes \sum_k \binom{n-j}{k} \alpha_2^{n-j+k} \otimes \mathrm{Sq}^{i-k} \mathrm{Sq}^j(\iota_n) \\
&= \textstyle\sum_{i,j,l} \binom{2n-j}{n+j-l} \alpha_1^{2n-i} \otimes \alpha_2^{2n-l} \otimes \mathrm{Sq}^{i+l-n-j} \mathrm{Sq}^j(\iota_n),
\end{aligned}$$

where we let $l = j - k + n$.

We prove that the last expression is invariant under a permutation of i and l. Let's assume this for now and derive the Adém relations. We then get

$$\sum_j \binom{n-j}{n+j-l} \mathrm{Sq}^{i+l-n-j} \mathrm{Sq}^j \iota_n = \sum_j \binom{n-j}{n+j-i} \mathrm{Sq}^{i+l-n-j} \mathrm{Sq}^j \iota_n. \tag{2}$$

We pick n such that most of the binomial coefficients on the left are 0. For integers r, s, let $n = 2^r - 1 + s$ and $l = n + s$. Then $\binom{n-j}{n+j-l} = \binom{2^r-1-(j-s)}{j-s}$. If r is sufficiently large and $j \neq s$ then there will be a factor $\binom{0}{1} = 0$ in the product formula of Lucas' theorem 2.6, so that the binomial coefficient is 0 mod 2. If $j = s$, then we get 1.

Therefore the equality in (2) becomes

$$\mathrm{Sq}^i \mathrm{Sq}^s \iota_n = \sum_j \binom{2^r + s - j - 1}{2^r + s - 1 + j - i} \mathrm{Sq}^{i+s-j} \mathrm{Sq}^j \iota_n = \sum_j \binom{2^r + s - j - 1}{i - 2j} \mathrm{Sq}^{i+s-j} \mathrm{Sq}^j \iota_n.$$

Then, if $i < 2s$ we have $\binom{2^r+s-j-1}{i-2j} = \binom{s-j-1}{i-2j}$. Indeed if $i < 2j$ both are zero, and if $i \geq 2j$ this means $j < s$ so that $s - j - 1 \geq 0$. Then for r large enough the 2^r term has a 2-adic expansion with only one 1, so contributes a factor $\binom{1}{0} = 1$, and hence can be ignored.

Thus, we have proved the Adém relations for certain values of n. But then by commutativity with suspension we have that it holds for all positive integers n, so for all fundamental classes of K_n, and so for all cohomology classes in spaces by naturality.

Now we need to explain and prove why iterating the squares lands in symmetric polynomials in $H^*(X)[\alpha_1, \alpha_2]$. To do this, we will consider some constructions from two perspectives.

The group action viewpoint:

We rephrase our original construction $X \to \Lambda X$. Equivalently, to construct ΛX, we can take $S^\infty \times X \times X$, identify all points with a coordinate equal to x_0, and then factor out by the diagonal $\mathbb{Z}_2$ action.

In the case of iterating the construction and obtaining the space $\Lambda(\Lambda X)$, we identify all points of $S^\infty \times (S^\infty \times X \times X) \times (S^\infty \times X \times X)$ with at least one x_0-coordinate. Now we need to factor out by some group action. This group G will be a subgroup of S_4, corresponding to the relevant permutations of our four X factors. We should have the transposition corresponding to the "diagonal action" with each S^∞ factor, and we will denote these as follows:

- $(13)(24) \in G$ such that $g(s_1, s_2, x_{11}, x_{12}, s_3, x_{21}, x_{22}) = (-s_1, s_3, x_{21}, x_{22}, s_2, x_{11}, x_{12})$,
- $(12) \in G$ such that $h_1(s_1, s_2, x_{11}, x_{12}, s_3, x_{21}, x_{22}) = (s_1, -s_2, x_{12}, x_{11}, s_3, x_{21}, x_{22})$,
- $(34) \in G$ such that $h_2(s_1, s_2, x_{11}, x_{12}, s_3, x_{21}, x_{22}) = (s_1, s_2, x_{11}, x_{12}, -s_3, x_{22}, x_{21})$.

These three transpositions generate the dihedral group $G = D_4 \subset S_4$. Note that this is the action of the wreath product $\mathbb{Z}_2 \wr \mathbb{Z}_2$, the group defined by the split exact sequence

$$0 \to \mathbb{Z}_2^2 \to \mathbb{Z}_2 \wr \mathbb{Z}_2 \to \mathbb{Z}_2 \to 0$$

Thus after identifying necessary points, $S^\infty \times (S^\infty \times X \times X) \times (S^\infty \times X \times X)$ is a D_4-space (a space equipped with a D_4 action).

We define a space $\Lambda_2 X$ with a $\mathbb{Z}_2 \times \mathbb{Z}_2$ action that maps into $\Lambda(\Lambda X)$. There is an action $\mathbb{Z}_2 \times \mathbb{Z}_2 \circlearrowright S^\infty \times S^\infty$, where the $\mathbb{Z}_2$ factors act on the corresponding S^∞ factors by sending a point to its antipode. One can also verify an action $\mathbb{Z}_2 \times \mathbb{Z}_2 \circlearrowright X^{2^2}$ by writing points of X^{2^2} as a 2×2 matrix and letting the first $\mathbb{Z}_2$ act by permuting the rows, and the second $\mathbb{Z}_2$ act by permuting the columns. We define $\Lambda_2 X$ by taking $S^\infty \times S^\infty \times X^{2^2}$, first collapsing the subspace of points having at least one X-coordinate equal to x_0, and then factoring out the diagonal $\mathbb{Z}_2 \times \mathbb{Z}_2$ action.

Now we define a map between these two spaces that passes to a map on the quotient spaces. Consider the map $u\colon S^\infty \times S^\infty \times X^{2^2} \to S^\infty \times (S^\infty \times X \times X)^2$ given by $(s, t, x_{11}, x_{12}, x_{21}, x_{22}) \mapsto (s, t, x_{11}, x_{12}, t, x_{21}, x_{22})$. For $y \in S^\infty \times S^\infty \times X^{2^2}$, and $\mathbb{Z}_2 \times \mathbb{Z}_2 \simeq \{(0,0), (1,0), (0,1), (1,1)\}$ acting on y, one can check that

$$\begin{aligned}
(0,0)y &= u(y), \\
(1,0)y &= (12)(34)(u(y)), \\
(0,1)y &= (13)(24)(u(y)), \\
(1,1)y &= (14)(23)(u(y)).
\end{aligned}$$

Therefore the orbit space of y is mapped to the orbit of $u(y)$ acted on by the $\mathbb{Z}_2 \times \mathbb{Z}_2$ subgroup of D_4 generated by $(12)(34)$ and $(13)(24)$. We can consider u as $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-equivariant, meaning it commutes with the $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-actions on each space. There is an induced map $u'\colon \Lambda_2(X) \to \Lambda(\Lambda X)$.

We have the embedding $\mathbb{R}P^\infty \wedge \mathbb{R}P^\infty \wedge X \to \Lambda(\Lambda X)$ is induced by $(s, t, x) \mapsto (s, t, x, x,) \mapsto (s, t, x, x, t, x, x)$. We can also consider the map $(s, t, x) \mapsto (s, t, x, x, x, x)$ which induces a map $\nabla_2\colon \mathbb{R}P^\infty \wedge \mathbb{R}P^\infty \wedge X \to \Lambda_2 X$. Importantly, we have $u(s, t, x, x, x, x) = u(s, t, x, x, t, x, x)$.

We will consider a $\mathbb{Z}_2$-action on $\mathbb{R}P^\infty \wedge \mathbb{R}P^\infty \wedge X$ and $\Lambda_2 X$ and show that ∇_2 is $\mathbb{Z}_2$-equivariant up to homotopy. We move on to the geometric approach to show that all this is enough to prove that we land in the symmetric polynomials in $H^*(X)[\alpha_1, \alpha_2]$.

The geometric viewpoint:

Let us describe our construction of $\Lambda_2 X$ in the geometric terms we first described the construction of the Steenrod squares in. We have the action on the smash product of spaces $\mathbb{Z}_2 \times \mathbb{Z}_2 \circlearrowright X^{\wedge 2^2}$ by permuting rows and columns as above. Therefore, $\mathbb{Z}_2 \times \mathbb{Z}_2 \circlearrowright S^\infty \times S^\infty \times X^{\wedge 2^2}$ by a diagonal action with quotient space $:= \Gamma_2 X$. We will claim that, similar to our original construction, this projects to $\mathbb{R}P^\infty \times \mathbb{R}P^\infty$ with a section, and we define $\Lambda_2 X$ by collapsing this section. The fibers of the projection $\Lambda_2 X \to \mathbb{R}P^\infty \times \mathbb{R}P^\infty$ are $X^{\wedge 2^2}$.

We are considering what happens to $\iota_n \in H^n(K_n)$, so we let $X = K_n$. We will also claim the following:

1. there is a map $\lambda_2\colon \Lambda_2 K_n \to K_{4n}$ such that the restriction to the fibres $K_n^{\wedge 2^2}$ is $\iota_n^{\otimes 2^2}$,
2. λ_2 is determined uniquely up to homotopy by the property that the restriction to the fibres $K_n^{\wedge 2^2}$ is $\iota_n^{\otimes 2^2}$.

The construction of this map is analogous to that of λ, and the uniqueness statement analogous to remark 2.12. The details can be seen in [1, Theorem 4L.13].

Similar to the case of ∇, the diagonal map $S^\infty \wedge S^\infty \wedge K_n \to S^\infty \wedge S^\infty \wedge K_n^{\wedge 2^2}$ given by $(s,t,x) \mapsto (s,t,x,x,x,x)$ induces a map $\nabla_2\colon \mathbb{R}P^\infty \wedge \mathbb{R}P^\infty \wedge K_n \to \Lambda_2 K_n$.

Then the following diagram commutes, where the cohomology class given by going along the bottom of the diagram is the one we already calculated:

$$\begin{tikzcd}
\mathbb{R}P^\infty \wedge \mathbb{R}P^\infty \wedge K_n \arrow[r, "\nabla_2"] \arrow[d, "\mathrm{Id}\wedge\nabla"] & \Lambda_2 K_n \arrow[r, "\lambda_2"] \arrow[d, "u'"] & K_{4n} \\
\mathbb{R}P^\infty \wedge \Lambda K_n \arrow[r, "\nabla"] & \Lambda(\Lambda K_n) \arrow[ur, "\lambda(\lambda)"'] &
\end{tikzcd}$$

Indeed, the square commutes by construction. Commutativity of the triangle up to homotopy follows once we check it commutes when restricted to the fibers $K_n^{\wedge 2^2}$. We have

$$\begin{tikzcd}
K_n^{\wedge 2^2} \arrow[r, "\iota_n^{\otimes 2^2}"] \arrow[d] & K_{4n} \\
(K_n^{\wedge 2})^{\wedge 2} \arrow[ur, "(\iota_n^{\otimes 2})^{\otimes 2}"'] &
\end{tikzcd}$$

and the downward map induces an isomorphism of cohomology since by Künneth $H^*(K_n^{\wedge 2^2}) \simeq (H^*(K_n))^{\otimes 2^2} \simeq (H^*(K_n)^{\otimes 2})^{\otimes 2} \simeq H^*((K_n^{\wedge 2})^{\wedge 2})$.

We show a symmetry of $\nabla_2^*\lambda_2^*$, which will yield a corresponding symmetry in the calculation of $(\mathrm{Id}\times\nabla)^*(\nabla^*(\lambda(\lambda)))$ by the commutativity of the above diagram up to homotopy. Consider the following commutative square:

$$\begin{tikzcd}
\mathbb{R}P^\infty \wedge \mathbb{R}P^\infty \wedge K_n \arrow[r, "\nabla_2"] \arrow[d, "\tau"] & \Lambda_2 K_n \arrow[d, "\tau"] \\
\mathbb{R}P^\infty \wedge \mathbb{R}P^\infty \wedge K_n \arrow[r, "\nabla_2"] & \Lambda_2 K_n
\end{tikzcd}$$

Each τ defines a $\mathbb{Z}_2$-action. The τ on the left permutes the two $\mathbb{R}P^\infty$ factors, and τ on the right is induced by switching the two S^∞ factors of $S^\infty \times S^\infty \times K_n^{\wedge 2^2}$, and interchanging the two subscripts in the tuples (x_{ij}) in $K_n^{\wedge 2^2}$.

Looking at the fiber in $\Lambda_2 K_n$, we have that τ is a transposition (swapping the off-diagonal), so $\iota_n^{\otimes 2^2} \mapsto (-1)^n \iota_n^{\otimes 2^2}$, so that $\tau^*\lambda_2^*(\iota_n) = (-1)^n \lambda_2^*(\iota_n)$. Then by commutativity of the diagram and uniqueness of maps when restricted to the fiber, we have $\tau^*\nabla_2^*\lambda_2^*(\iota_n = (-1)^n \nabla_2^*\lambda_2^*(\iota_n)$.

Note that we can write $\nabla_2^*\lambda_2^*(\iota_n) = \sum_{r,s} \alpha_r \times \alpha_s \times \varphi_{rs}$ for $\alpha_r, \alpha_s \in H^*(\mathbb{R}P^\infty)$ and $\varphi_{rs} \in H^*(K_n)$. Then $\tau^*\nabla_2^*\lambda_2^*(\iota_n) = \sum_{r,s} (-1)^{rs+n} \alpha_s \times \alpha_r \times \varphi_{rs}$ by commutativity of cross product. Therefore $\varphi_{rs} = (-1)^{rs+n}\varphi_{sr}$, which are equal for cohomology with coefficients in $\mathbb{Z}_2$.

Looking back to our expression

$$(\mathrm{Id}\wedge\nabla)^*(\nabla^*(\lambda(\lambda))) = \sum_{i,j,l} \binom{2n-j}{n+j-l} \alpha^{2n-i} \otimes \alpha^{2n-l} \otimes \mathrm{Sq}^{i+l-n-j}\,\mathrm{Sq}^j(\iota_n)$$

we see that this is invariant under switching i and l, as desired.

3 Steenrod algebra and the dual Steenrod algebra

By considering the compositions of the squares, we can form a graded algebra generated by the $\{\mathrm{Sq}^i\}_{i\geq 0}$, where Sq^i has degree i. The Adém relations tell us that some of the elements of this algebra are equal. The natural thing to do is to quotient out by these relations, which is what we call the Steenrod algebra $\mathcal{A}$.

We say a composition of squares is admissible if we cannot apply any Adém relation to it. Given any composition $\mathrm{Sq}^{i_1}\mathrm{Sq}^{i_2}\cdots\mathrm{Sq}^{i_r}$, we can keep applying Adém relations until we have expressed $\mathrm{Sq}^{i_1}\mathrm{Sq}^{i_2}\cdots\mathrm{Sq}^{i_r}$ as a sum of admissibles (we will show this process terminates later). For example, consider the different ways we can apply Adém relations in the following:

$$\begin{tikzcd}
 & \mathrm{Sq}^1\mathrm{Sq}^2\mathrm{Sq}^3 \arrow[dl] \arrow[dr] & \\
\mathrm{Sq}^3\mathrm{Sq}^3 \arrow[d] & & \mathrm{Sq}^1(\mathrm{Sq}^4\mathrm{Sq}^1+\mathrm{Sq}^5) \arrow[d] \\
\mathrm{Sq}^5\mathrm{Sq}^1 & & \mathrm{Sq}^5\mathrm{Sq}^1+0
\end{tikzcd}$$

Reassuringly, we get $\mathrm{Sq}^5\mathrm{Sq}^1$ regardless of what order we apply the Adém relations. But we don't want to have to verify that in every instance. To verify that the order in which we apply the Adém relations does not matter, we need to show that there are no relations between the admissibles in $\mathcal{A}$.

Of course, we should really impose all relations that exist to get to what we would call the "Steenrod algebra". Showing that there are no relations between the admissibles ensures also that imposing the Adém relations is enough. Indeed, if this is true then we're not going to come up with some construction like that of section 2.3 and show that $\mathrm{Sq}^4=\mathrm{Sq}^2\mathrm{Sq}^2$; this would be saying the admissibles Sq^4 and $\mathrm{Sq}^3\mathrm{Sq}^1$ are equal.

Now we know what we need to do, and the strategy is to show that the admissible squares act "linearly independently" on the cohomology of a certain space. This leads us to the Serre-Cartan basis of $\mathcal{A}$.

After this, we turn to looking at the dual Steenrod algebra.

3.1 Steenrod algebra and Serre-Cartan basis

Consider the graded module $M=(M_i)_i$, where each M_i is isomorphic to $\mathbb{Z}_2$, and generated by Sq^i (so that Sq^i has degree i). Let $\Gamma(M)$ be the tensor algebra of M with multiplication $\varphi\colon\Gamma(M)\otimes\Gamma(M)\to\Gamma(M)$ being the composition of squares.

Consider the two-sided ideal J generated by $\mathrm{Sq}^0+\mathrm{Id}$ and the Adém relations for $a<2b$:

$$Q(a,b)=\mathrm{Sq}^a\mathrm{Sq}^b+\sum_{c\geq 0}\binom{b-c-1}{a-2c}\mathrm{Sq}^{a+b-c}\mathrm{Sq}^c$$

We quotient $\Gamma(M)$ by this ideal to obtain the Steenrod algebra.

Definition 3.1 (Steenrod algebra)**.** The Steenrod algebra $\mathcal{A}$ is the graded algebra on generators $\{\mathrm{Sq}^i\}_{i\geq 0}$ where we impose the Adém relations.

We introduce the notation that for $I=\{i_1,\dots i_r\}$ with $i_j\geq 0$ for all j, Sq^I denotes the composition of squares $\mathrm{Sq}^{i_1}\mathrm{Sq}^{i_2}\cdots\mathrm{Sq}^{i_r}$.

The following definitions will be useful to us when talking about the compositions of the squares. Note that the definition of the excess of I describes how much Sq^I deviates from being inadmissible.

Definition 3.2. Let $I=\{i_1,\dots i_r\}$ be a sequence with $i_j\geq 0$ for all j.

1. The length of I is $l(I) = r$.
2. The moment of I is $m(I) = \sum_{j=1}^{r} j \cdot i_j$.
3. The degree of I is $|I| = \sum_{j=1}^{r} i_j$.
4. We call I admissible if $i_j \geq 2i_{j+1}$ for all $1 \leq j \leq r-1$. Otherwise, we say I is inadmissible.
5. The excess of I is $e(I) = \sum_{j=1}^{r-1}(i_j - 2i_{j+1}) + i_r = 2i_1 - \sum_{j=1}^{r} i_j$.

It is easy to describe how $\mathcal{A}$ acts on $H^1(\mathbb{R}P^\infty)$, and we will use this to our advantage multiple times. We have:

Lemma 3.3. *Let $\alpha \in H^1(\mathbb{R}P^\infty)$ be a generator, and define the sequences $I_k = \{2^{k-1}, \ldots, 2, 1\}$. Then,*

1. *$\mathrm{Sq}^I(\alpha) = \alpha^{2^k}$ if $I = I_k$, and $\mathrm{Sq}^I(\alpha) = 0$ for all other non-zero admissible sequences I,*
2. *For any non-zero I, $\mathrm{Sq}^I(\alpha) = 0$ unless I is obtained from I_k by interspersion of zeroes.*

Proof. Follows from example 2.8 which says that

$$\mathrm{Sq}^i(\alpha^{2^l}) = \begin{cases} \alpha^{2^l} & \text{for } i = 0, \\ \alpha^{2^{l+1}} & \text{for } i = 2^l, \\ 0 & \text{otherwise.} \end{cases}$$

□

Given a space X, we have the left-action $\mathcal{A} \otimes H^*(X) \to H^*(X)$. Using a particular space and cohomology class, we show linear independence of the admissible squares over $\mathbb{Z}_2$.

Proposition 3.4 ([3, Proposition 3.2])**.** *Let R_n be the n-fold product of $\mathbb{R}P^\infty$, with $H^*(R_n) = \mathbb{Z}_2[\alpha_1, \ldots, \alpha_n]$ for α_i classes of degree 1 such that the i-th copy of R_n has $H^*(\mathbb{R}P^\infty) \simeq \mathbb{Z}_2[\alpha_i]$. Let w be the product $w = \alpha_1\alpha_2\cdots\alpha_n$.*

Then the map $\theta\colon \mathcal{A} \to H^*(R_n)$ defined by evaluation on w sends the Sq^I of degree $\leq n$ and I admissible into linearly independent elements.

Proof. We prove by induction on n. The case for $n = 1$ is clear since $\mathrm{Sq}^0 \alpha = \alpha$ and $\mathrm{Sq}^1 \alpha = \alpha^2$, which are linearly independent over $\mathbb{Z}_2$.

We just need to check the linear independence for admissible sequences of the same degree. Suppose $\sum a_I \mathrm{Sq}^I w = 0$ where the sum is taken over admissible sequences I of some degree $q \leq n$. We prove that all $a_I = 0$ by a decreasing induction on the length $l(I)$ of I. Suppose that $a_I = 0$ for any admissible sequence of degree q and $l(I) > m$. Our expression thus reduces to as $\sum_{l(I)=m} a_I \mathrm{Sq}^I w + \sum_{l(I)<m} a_I \mathrm{Sq}^I w$. This expression is a class in the $(q+n)$-th cohomology group, and the Künneth formula gives $H^{q+n}(R_n) = \sum_s H^s(\mathbb{R}P^\infty) \otimes H^{q+n-s}(R_{n-1})$.

Let g denote the projection onto the summand for $s = 2^m$. By the Cartan formula, we have $\mathrm{Sq}^I w = \mathrm{Sq}^I(\alpha_1 \times \alpha_2\alpha_3\cdots\alpha_n) = \sum_{J \leq I} \mathrm{Sq}^J \alpha_1 \times \mathrm{Sq}^{I-J} \alpha_2 \cdots \alpha_n$. By lemma 3.3 we see that for $\mathrm{Sq}^J \alpha_1$ to be non-zero and of degree s, we need $J = I_m = \{2^{m-1}, \ldots 2, 1\}$ and so $g \mathrm{Sq}^I w$ can only be non-zero for $l(I) = m$:

$$g \mathrm{Sq}^I w = \begin{cases} 0 & \text{if } l(I) < m, \\ \alpha_1^{2^m} \otimes \mathrm{Sq}^{I-I_m} \alpha_2 \cdots \alpha_n & \text{if } l(I) = m. \end{cases}$$

Therefore we have $g\left(\sum_{l(I)=m} a_I \mathrm{Sq}^I w + \sum_{l(I)<m} a_I \mathrm{Sq}^I w\right) = \alpha_1^{2^m} \otimes \sum_{l(I)=m} a_I(\mathrm{Sq}^{I-I_m} \alpha_2 \cdots \alpha_n)$ $= 0$. We have that for all I admissible of length m, $I - I_m$ is admissible of length $\leq m$ and degree $q - 2^m + 1$, and the correspondence of I and $I - I_m$ is one-to-one. Then $q - 2^m + 1 \leq n - 1$ so the inductive hypothesis on n implies that all a_I in our previous expression are 0. Thus $a_I = 0$ for $l(I) = m$. □

Theorem 3.5 (Serre-Cartan basis)**.** *The Steenrod algebra $\mathcal{A}$ has $\{\mathrm{Sq}^I\colon I \text{ admissible}\}$ as a basis.*

Proof.

1. Spanning set: Clearly monomials of the form Sq^I generate $\mathcal{A}$. We prove that the admissible monomials are also generators by a downward induction on the moment $m(I)$ of a sequence I. If $I = \{i_1, i_2, \ldots, i_r\}$ is not admissible, i.e. there exists s such that $i_s < 2i_{s+1}$, then we can rewrite Sq^I by replacing $\mathrm{Sq}^{i_s}\mathrm{Sq}^{i_{s+1}}$ with the corresponding Adém relation. This replacement strictly lowers the moment of I, so by induction on the moment it follows that Sq^I can be expressed as the sum of admissible monomials.

2. Linear independence: This is a consequence of proposition 3.4. For any set of admissible squares, we can take n large enough so that they are mapped into linearly independent classes over $\mathbb{Z}_2$ in $H^*(R_n)$.

$\square$

Corollary 3.6. *The mapping $\theta\colon \mathcal{A} \to H^*(R_n)$ given by evaluation on w as in proposition 3.4 is a monomorphism in degrees $\leq n$.*

Example 3.7. The Serre-Cartan basis up to degree 9:

0	Sq^0				
1	Sq^1				
2	Sq^2				
3	Sq^3	$\mathrm{Sq}^2\mathrm{Sq}^1$			
4	Sq^4	$\mathrm{Sq}^3\mathrm{Sq}^1$			
5	Sq^5	$\mathrm{Sq}^4\mathrm{Sq}^1$			
6	Sq^6	$\mathrm{Sq}^5\mathrm{Sq}^1$	$\mathrm{Sq}^4\mathrm{Sq}^2$		
7	Sq^7	$\mathrm{Sq}^6\mathrm{Sq}^1$	$\mathrm{Sq}^5\mathrm{Sq}^2$	$\mathrm{Sq}^4\mathrm{Sq}^2\mathrm{Sq}^1$	
8	Sq^8	$\mathrm{Sq}^7\mathrm{Sq}^1$	$\mathrm{Sq}^6\mathrm{Sq}^2$	$\mathrm{Sq}^5\mathrm{Sq}^2\mathrm{Sq}^1$	
9	Sq^9	$\mathrm{Sq}^8\mathrm{Sq}^1$	$\mathrm{Sq}^7\mathrm{Sq}^2$	$\mathrm{Sq}^6\mathrm{Sq}^2\mathrm{Sq}^1$	$\mathrm{Sq}^6\mathrm{Sq}^3$

Example 3.8. Consider the subalgebra $\mathcal{A}(2) = \langle \mathrm{Sq}^0, \mathrm{Sq}^1, \mathrm{Sq}^2 \rangle$ of $\mathcal{A}$ generated by Sq^0, Sq^1 and Sq^2. This is a vector space of dimension 8 over $\mathbb{Z}_2$, whose relations we can describe as follows:

$$\begin{tikzcd}
 & \mathrm{Sq}^1 \arrow[rr, dash] & & \mathrm{Sq}^{\{2,1\}} \arrow[dr, dash] & & & \\
\mathrm{Sq}^0 \arrow[ur, dash] \arrow[rr, dash] & & \mathrm{Sq}^2 \arrow[rr, dash] \arrow[dr, dash] & & \mathrm{Sq}^{\{2,2\}} = \mathrm{Sq}^{\{3,1\}} \arrow[rr, dash] & & \mathrm{Sq}^{\{2,2,2\}} = \mathrm{Sq}^{\{5,1\}} \\
 & & & \mathrm{Sq}^{\{1,2\}} = \mathrm{Sq}^3 \arrow[rr, dash] & & \mathrm{Sq}^{\{2,1,2\}} = \mathrm{Sq}^{\{4,1\}} + \mathrm{Sq}^5 \arrow[ur, dash] & \\
0 & 1 & 2 & 3 & 4 & 5 & 6
\end{tikzcd}$$

The numbers at the bottom are the degrees. The shorter diagonal lines denote multiplication by Sq^1 on the left, and the longer straight lines denote multiplication by Sq^2 on the left.

- An Adém relation gives $\mathrm{Sq}^1\mathrm{Sq}^1 = 0$. That is also clear since Sq^1 is a Bockstein homomorphism. We have $\mathrm{Sq}^1\mathrm{Sq}^2 = \mathrm{Sq}^3$ by the Adém relations.

- We can apply an Adém relation to $\mathrm{Sq}^2\mathrm{Sq}^2$ and get $\mathrm{Sq}^2\mathrm{Sq}^2 = \mathrm{Sq}^3\mathrm{Sq}^1 = \mathrm{Sq}^1\mathrm{Sq}^2\mathrm{Sq}^1$. From this we see $\mathrm{Sq}^1\mathrm{Sq}^2\mathrm{Sq}^2 = \mathrm{Sq}^1\mathrm{Sq}^1\mathrm{Sq}^2\mathrm{Sq}^1 = 0$. Likewise $\mathrm{Sq}^2\mathrm{Sq}^2\mathrm{Sq}^1 = 0$.

- The only element of the algebra of degree 5 is $\mathrm{Sq}^2\mathrm{Sq}^1\mathrm{Sq}^2$. Then $\mathrm{Sq}^1\mathrm{Sq}^2\mathrm{Sq}^1\mathrm{Sq}^2 = \mathrm{Sq}^2\mathrm{Sq}^2\mathrm{Sq}^2$.

- Finally, we show that the diagram cannot be extended further to the right. We have $\mathrm{Sq}^1\mathrm{Sq}^2\mathrm{Sq}^2\mathrm{Sq}^2 = 0$ since $\mathrm{Sq}^1\mathrm{Sq}^2\mathrm{Sq}^2 = 0$, and $\mathrm{Sq}^2\mathrm{Sq}^2\mathrm{Sq}^2\mathrm{Sq}^2 = \mathrm{Sq}^1\mathrm{Sq}^2\mathrm{Sq}^1\mathrm{Sq}^1\mathrm{Sq}^2\mathrm{Sq}^1 = 0$.

Note that the inclusion $\mathcal{A}(2) \to \mathcal{A}$ is an isomorphism up to degree 3. Sq^4 cannot be expressed in terms of $\mathrm{Sq}^1, \mathrm{Sq}^2$, and is in fact indecomposable.

Definition 3.9 (Decomposables and indecomposables)**.** Let $\overline{\mathcal{A}}$ be the ideal of $\mathcal{A}$ consisting of elements of degree > 0. Let $\varphi\colon \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ be the multiplication map of $\mathcal{A}$.

1. The set of decomposables of $\mathcal{A}$ is the image of $\overline{\mathcal{A}} \otimes \overline{\mathcal{A}}$ under φ (this is a two sided ideal),
2. The set of indecomposables of $\mathcal{A}$ is $\overline{\mathcal{A}}/\varphi(\overline{\mathcal{A}} \otimes \overline{\mathcal{A}})$.

The only elements of $\mathcal{A}$ that are possibly indecomposable are $\{\mathrm{Sq}^i\}_{i>0}$. Then Sq^i is decomposable if $\mathrm{Sq}^i = \sum_{t<i} a_t \,\mathrm{Sq}^t$ where each a_t is a sequence of squares, and indecomposable if no such relation exists. For example, Sq^6 is decomposable because of the Adém relation $\mathrm{Sq}^2 \mathrm{Sq}^4 = \mathrm{Sq}^6 + \mathrm{Sq}^5 \mathrm{Sq}^1$.

Theorem 3.10. Sq^i *is indecomposable if and only if i is a power of* 2.

Proof. Suppose that i is a power of 2. We show that Sq^i is indecomposable by choosing a cohomology class x such that $\mathrm{Sq}^t(x)$ is only non-trivial for $t = 0, i$. Consider α the generator of $H^1(\mathbb{R}P^\infty)$, so that $H^*(\mathbb{R}P^\infty) = \mathbb{Z}_2[\alpha]$. Then $\mathrm{Sq}^i(\alpha^i)$ cannot be written as a sum of lower squares applied to α^i. Indeed by example 2.8, $\mathrm{Sq}^0(\alpha^i) = \alpha^i$, $\mathrm{Sq}^i(\alpha^i) = \alpha^{2i}$, and $\mathrm{Sq}^t \alpha^i = 0$ for all other t.

Conversely, we show that Sq^i is decomposable for $i = a + b$ with $b = 2^k$ and $0 < a < 2^k$. Then $a < 2b$, so we have the Adém relation

$$\mathrm{Sq}^a \mathrm{Sq}^b = \sum_{c\geq 0} \binom{b-c-1}{a-2c} \mathrm{Sq}^{a+b-c} \mathrm{Sq}^c = \binom{b-1}{a} \mathrm{Sq}^{a+b} + \sum_{c>0} \binom{b-c-1}{a-2c} \mathrm{Sq}^{a+b-c} \mathrm{Sq}^c,$$

and $\binom{b-1}{a} = 1$ by Lucas' theorem 2.6 , so that Sq^{a+b} is indecomposable. □

We can use the previous result to show that if n is not a power of 2, then there does not exist a space X such that $H^*(X; \mathbb{Z}_2) \simeq \mathbb{Z}_2[x]$ with $|x| = n$.

Corollary 3.11. *If X is a space with polynomial $\mathbb{Z}_2$-cohomology $H^*(X) \simeq \mathbb{Z}_2[x]$ and $x^2 \neq 0$, then $|x| = 2^k$ for some k.*

Proof. Let $|x| = i$. Then $\mathrm{Sq}^i(x) = x^2$. Then all squares of lower, positive degree are zero on x, hence Sq^i is indecomposable, hence i is a power of 2. □

The above result can be improved. J.F. Adams shows that the only possibilities are $k = 0, 1, 2, 3$.

3.2 Dual Steenrod algebra and Milnor basis

Next, we will study the structure of the dual Steenrod algebra. Milnor noticed that $\mathcal{A}$ has a commutative co-product, giving it the structure of a Hopf algebra. When we consider the dual Steenrod algebra $\mathcal{A}^*$, we therefore get that it is equipped with a commutative product (given by the dual of the co-product of $\mathcal{A}$). In fact, the structure of the dual Steenrod algerba turns out to be a lot nicer than that of $\mathcal{A}$; it is a polynomial algebra.

3.2.1 Steenrod algebra as a Hopf algebra

Let R be a commutative ring with unit. We will use the following definitions described by Milnor and Moore in [4].

Definition 3.12 (R-algebra)**.** An algebra A over R is a graded R-module, that is $A = (A_i)_i$, with a multiplication morphism of graded R-modules $\varphi\colon A \otimes A \to A$ and a map $\eta\colon R \to A$ such that the following two diagrams are commutative:

$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "1\otimes\varphi"] \arrow[d, "\varphi\otimes 1"] & A \otimes A \arrow[d, "\varphi"] \\ A \otimes A \arrow[r, "\varphi"] & A \end{tikzcd} \qquad \begin{tikzcd} & A \otimes R \arrow[dr, "1\otimes\eta"] & & \\ A \arrow[ur, "\simeq"] \arrow[dr, "\simeq"'] & & A \otimes A \arrow[r, "\varphi"] & A \\ & R \otimes A \arrow[ur, "\eta\otimes 1"'] & & \end{tikzcd}$$

where the first describes associativity of φ and the second implies η is a unit of the multiplication.

Definition 3.13 (R-coalgebra)**.** A coalgebra A over R is a graded R-module with a comultiplication morphism of graded R-modules $\psi\colon A \to A \otimes A$ and a map $\varepsilon\colon A \to R$ such that the following two diagrams are commutative:

$$
\begin{tikzcd}
A \arrow[r, "\psi"] \arrow[d, "\psi"] & A \otimes A \arrow[d, "\psi\otimes 1"] \\
A \otimes A \arrow[r, "1\otimes\psi"] & A \otimes A \otimes A
\end{tikzcd}
\qquad
\begin{tikzcd}
 & A \otimes R \arrow[dl, "\simeq"'] & & \\
A & & A \otimes A \arrow[ul, "1\otimes\varepsilon"'] \arrow[dl, "\varepsilon\otimes 1"] & A \arrow[l, "\psi"] \\
 & R \otimes A \arrow[ul, "\simeq"] & &
\end{tikzcd}
$$

where the first describes coassociativity of ψ and the second implies ε is a counit of the comultiplication.

Definition 3.14 (Hopf algebra)**.** A Hopf algebra over R is a graded R-module A with morphisms of graded R-modules

$$
\begin{array}{ll}
\varphi\colon A \otimes A \to A & \eta\colon R \to A \\
\psi\colon A \to A \otimes A & \varepsilon\colon A \to R
\end{array}
$$

such that

1. (A, φ, η) is an algebra over R with augmentation ε (meaning ε is a morphism of algebras, consider R as an alegbra over itself),
2. (A, ψ, ε) is a coalgebra over R with augmentation η (considering R as a colagebra),
3. The following diagram commutes

$$
\begin{tikzcd}
A \otimes A \arrow[r, "\varphi"] \arrow[d, "\psi\otimes\psi"] & A \arrow[r, "\psi"] & A \otimes A \\
A \otimes A \otimes A \otimes A \arrow[rr, "1\otimes T\otimes 1"] & & A \otimes A \otimes A \otimes A \arrow[u, "\varphi\otimes\varphi"']
\end{tikzcd}
$$

where T is the twist homomorphism mapping $x \otimes y \mapsto (-1)^{|x||y|} y \otimes x$.

The Steenrod algebra is a $\mathbb{Z}_2$-algebra, with multiplcation φ given by the composition of the squares, which is associative. Moreover $\mathcal{A}$ is augmented: there is an augmentation map $\epsilon\colon \mathcal{A} \to \mathbb{Z}_2$ sending Sq^0 to 1, and $\ker \epsilon$ consists of all squares of positive degree. Therefore the restriction of ε to the degree 0 part of $\mathcal{A}$, ϵ_0, is an isomorphism, and so we say $\mathcal{A}$ is connected. Also, $\epsilon_0^{-1}\colon \mathbb{Z}_2 \to \mathcal{A}$ is the unit of multiplcation.

In [5], Milnor shows that the Steenrod algebra $\mathcal{A}$ is a Hopf algebra. The diagonal map is induced by the Cartan formula for the squares, and the counit is ϵ_0. The cohomology of a space $H^*(X)$ is an $\mathcal{A}$-module since $\mathcal{A}$ acts on $H^*(X)$ on the left, so we have a map $\mathcal{A} \otimes H^*(X) \to H^*(X)$. Let $\alpha\colon H^*(X) \otimes H^*(X) \to H^*(X \times X)$ be the isomorphism given by the Künneth theorem. Then $H^*(X \times X)$ has its usual $\mathcal{A}$-module strucure. Milnor's idea was to show that there is a co-product $\psi\colon \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ such that we have another equivalent $\mathcal{A}$-module structure on $H^*(X \times X)$, given as

$$
\mathcal{A} \otimes H^*(X \times X) \xrightarrow{\psi\otimes\alpha^{-1}} \mathcal{A} \otimes \mathcal{A} \otimes H^*(X) \otimes H^*(X) \to H^*(X) \otimes H^*(X) \xrightarrow{\alpha} H^*(X \times X),
$$

where $\mathcal{A} \otimes \mathcal{A}$ acts on $H^*(X) \otimes H^*(X)$ by $(\sum_i a_i' \otimes a_i'')(\alpha \otimes \beta) = \sum_i a_i'(\alpha) \otimes a_i''(\beta)$.

Proposition 3.15 ([5, Lemma 1])**.** *For each $a \in \mathcal{A}$, there is a unique element $\psi(a) = \sum_i a_i' \otimes a_i''$ of $\mathcal{A} \otimes \mathcal{A}$ such that*

$$
ac(\alpha \otimes \beta) = c\psi(a)(\alpha \otimes \beta)
$$

is satisfied for all X and α, $\beta \in H^*(X)$, where $c\colon H^*(X) \otimes H^*(X) \to H^*(X)$ is the cup product. Furthermore, $\psi\colon \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ is a ring homomorphism.

Sketch proof. We will just show existence. We define ψ on the generators of $\Gamma(M)$ (the Steenrod algebra before quotienting out by the Adém relations) as $\psi(\mathrm{Sq}^i) = \sum_j \mathrm{Sq}^j \otimes \mathrm{Sq}^{i-j}$, and require that this is an algebra homomorphism. We need to prove that this passes to a map $\psi\colon \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$. Firstly, it extends to a map $\psi\colon \Gamma(M) \to \mathcal{A} \otimes \mathcal{A}$. Let $p\colon \Gamma(M) \to \mathcal{A}$ be the projection to the quotient. We show that $\ker p \subset \ker \psi$. To do so, consider the following set-up:

$$\begin{tikzcd}
\Gamma(M) \arrow[r, "p"] \arrow[d, "\psi"] & \mathcal{A} \arrow[drr, "\theta'"] & & \\
\mathcal{A} \otimes \mathcal{A} \arrow[r, "\theta\otimes\theta"] & H^*(R_n) \otimes H^*(R_n) \arrow[r, "\alpha"] & H^*(R_n \times R_n) = H^*(R_{2n}) &
\end{tikzcd}$$

The map θ is that which is defined in propositon 3.4, and θ' defined analogously for $2n$. Then by corollary 3.6, we have that θ is monomorphic up to degree n, and θ' is monomorphic up to degree n.

The diagram commutes on elements Sq^i, as $\alpha(\theta \otimes \theta)(\psi(\mathrm{Sq}^i)) = \alpha(\theta \otimes \theta)(\sum_j \mathrm{Sq}^j \otimes \mathrm{Sq}^{i-j}) = \alpha(\sum_j \mathrm{Sq}^j(\alpha_1 \cdots \alpha_n) \otimes \mathrm{Sq}^{i-j}(\alpha'_1 \cdots \alpha'_n)) = \sum_j \mathrm{Sq}^j(\alpha_1 \cdots \alpha_n) \times \mathrm{Sq}^{i-j}(\alpha'_1 \cdots \alpha'_n) = \mathrm{Sq}^i(\alpha_1 \cdots\ \alpha_n \cdot \alpha_{n+1} \cdots \alpha_{n+2})$ with the last step being the Cartan formula, and $\theta'(p(\mathrm{Sq}^i)) = \theta'(\mathrm{Sq}^i) = \mathrm{Sq}^i(\alpha_1 \cdots \alpha_n)$. One can then show it is commutative in general.

Then for $z \in \Gamma(M)$ with $p(z) = 0$, we can choose n large enough so that $\theta \otimes \theta$ is injective on $\psi(z)$, and so $\theta'(p(z)) = 0 = \alpha(\theta \otimes \theta)(\psi(z))$ implies $\psi(z) = 0$. $\square$

Example 3.16. We have $\psi(\mathrm{Sq}^3) = 1 \otimes \mathrm{Sq}^3 + \mathrm{Sq}^1 \otimes \mathrm{Sq}^2 + \mathrm{Sq}^2 \otimes \mathrm{Sq}^1 + \mathrm{Sq}^3 \otimes 1$.

For $\alpha \in H^1(\mathbb{R}P^\infty)$, we have $\mathrm{Sq}^3(\alpha^3) = \alpha^6$, and $\psi(\mathrm{Sq}^3)(\alpha \otimes \alpha^2) = \alpha \otimes \mathrm{Sq}^3 \alpha^2 + \mathrm{Sq}^1 \alpha \otimes \mathrm{Sq}^2 \alpha^2 + \mathrm{Sq}^2 \alpha \otimes \mathrm{Sq}^1 \alpha^2 + \mathrm{Sq}^3 \alpha \otimes \alpha^2 = \mathrm{Sq}^1 \alpha \otimes \mathrm{Sq}^2 \alpha^2 = \alpha^2 \otimes \alpha^4$.

Corollary 3.17 ([5, Theorem 1])**.** *$\mathcal{A}$ is a Hopf algebra.*

The dual of a Hopf algebra is also a Hopf algebra. Therefore we have

$$\mathcal{A} \xrightarrow{\psi} \mathcal{A} \otimes \mathcal{A} \xrightarrow{\varphi} \mathcal{A},$$

and the dual

$$\mathcal{A}^* \xrightarrow{\varphi^*} \mathcal{A}^* \otimes \mathcal{A}^* \xrightarrow{\psi^*} \mathcal{A}^*$$

are Hopf algebras. $\mathcal{A}$ is an algebra with associative, but non-commutative product. However ψ, being induced by the Cartan formula, can be shown to be associative and commutative. Therefore $\mathcal{A}^*$ has a commutative product, which makes its study easier.

Note that when we say commutative, we mean commutative with respect to the twist homomorphism T given in the Hopf algebra definitions. Since we are working over $\mathbb{Z}_2$, we don't need to worry about the (-1) factor.

3.2.2 Basis of the dual Steenrod algebra

Recall from lemma 3.3 that $\mathrm{Sq}^I(\alpha)$ is non-zero for $I = I_k$ with $|I_k| = 2^k - 1$, and $\alpha \in H^1(\mathbb{R}P^\infty)$ the generator.

Definition 3.18. For each $k \geq 0$, let $\xi_k \in \mathcal{A}^*_{2^k-1}$ be characterised by $\xi_k(\theta)(\alpha^{2^k}) = \theta(\alpha) \in H^{2^k}(\mathbb{R}P^\infty)$ for all $\theta \in \mathcal{A}_{2^k-1}$. Denote $\xi_k(\theta) \equiv \langle \xi_k, \theta \rangle$.

Example 3.19. ξ_0 is the unit of $\mathcal{A}^*$, with $\xi_0(\mathrm{Sq}^0) = 1$, and 0 otherwise. ξ_2 is the degree 3 element in $\mathcal{A}^*$ with $\xi_2(\theta)(\alpha^4) = \theta(\alpha)$, for $\theta \in \mathcal{A}$ of degree 3. For $\theta = \mathrm{Sq}^3 = \mathrm{Sq}^1 \mathrm{Sq}^2$, we have $\mathrm{Sq}^3 \alpha = 0$, so $\xi_2(\mathrm{Sq}^3) = 0$. For $\theta = \mathrm{Sq}^2 \mathrm{Sq}^1$, $\mathrm{Sq}^2 \mathrm{Sq}^1 \alpha = \alpha^4$, so $\xi_2(\mathrm{Sq}^2 \mathrm{Sq}^1) = 1$.

Proposition 3.20. *Let I be admissible. For $k \geq 1$, $\langle \xi_k, \mathrm{Sq}^I \rangle = 1$ if $I = I_k$ (i.e. ξ_k is the dual element to Sq^{I_k}). Otherwise $\langle \xi_k, \mathrm{Sq}^I \rangle = 0$. For arbitrary I, $\langle \xi_k, \mathrm{Sq}^I \rangle = 0$, unless I is obtained from some I_k by interspersion of zeroes.*

Proof. Direct consequence of lemma 3.3. □

Let $\mathcal{R}$ denote the set of all finite sequences of non-negative integers. Let $\mathcal{J} \subset \mathcal{R}$ be the subset of admissible sequences.

Define an isomorphism $\gamma\colon \mathcal{J} \to \mathcal{R}$ by $\gamma(\{a_1, \ldots a_k\}) = \{a_1 - 2a_2, a_2 - 2a_3, \ldots, a_k\}$, which makes sense since $\{a_1, \ldots a_k\}$ admissible means $a_i \geq 2a_{i+1}$ for all i.

Definition 3.21. For each $R = \{r_1, r_2, \ldots r_k\} \in \mathcal{R}$, we define $\xi^R \in \mathcal{A}^*$ by

$$\xi^R = \prod_{i=1}^{k} (\xi_i)^{r_i}.$$

Note that for $I = \{i_1, i_2, \ldots i_k\} \in \mathcal{J}$ the degree of Sq^I is exactly the degree of $\xi^{\gamma(I)}$, since

$$|\xi^{\gamma(I)}| = |\xi_k^{i_k} \prod_{j=1}^{k-1} (\xi_j)^{i_j - 2i_{j+1}}| = i_k(2^k - 1) + \sum_{j=1}^{k-1} (i_j - 2i_{j+1})(2^j - 1) =$$

$$i_k(2^k-1) + \sum_{j=1}^{k-1} (2^j i_j - 2^{j+1} i_{j+1}) + \sum_{j=1}^{k-1} (-i_j + 2i_{j+1}) = -i_k + 2i_k + 2i_1 - i_1 + \sum_{j=2}^{k-1} i_j = \sum_{j=1}^{k} i_j = |\,\mathrm{Sq}^I\,|.$$

We order the sequences of $\mathcal{J}$ lexicographically from the right, for example:

Example 3.22. $\{8, 4, 2, 0, \ldots\} > \{9, 4, 1, 0, \ldots\} > \{9, 4, 0, \ldots\} > \{17, 3, 0, \ldots\}$.

Milnor considers for each degree n, the matrix with entries $\langle \xi^R, Sq^I \rangle$, where I ranges over admissible sequences of degree n and ξ^R ranges over all ξ^R of degree n. It follows that we range over all $\xi^{\gamma(J)}$ as J ranges over all admissible sequences of degree n, since γ is an isomorphism and the degrees check out as explained above. He shows that this matrix is lower triangular with 1s on the diagonal, hence invertible. Such admissible sequences I form a basis for the degree n part of $\mathcal{A}$, and so the ξ^R form a basis for the degree n part of $\mathcal{A}^*$. The proof is as follows:

Theorem 3.23 ([5, Lemma 8])**.** *For $I, J \in \mathcal{J}$, $\langle \xi^{\gamma(J)}, \mathrm{Sq}^I \rangle = 0$ for $I < J$. If $I = J$, $\langle \xi^{\gamma(J)}, \mathrm{Sq}^I \rangle =$ 1.*

Proof. Suppose that $J = I_k = \{2^{k-1}, \ldots, 2, 1\}$. Then $\gamma(J)$ has a 1 in the k-th position, and zeroes elsewhere. Then $\langle \xi^{\gamma(J)}, \mathrm{Sq}^I \rangle = \langle \xi_k, \mathrm{Sq}^I \rangle$, which is non-zero for $I = I_k = J$ by proposition 3.20.

More generally, we prove by downward induction. For $J = 0$, there is nothing to prove.

Let $J = \{a_1, \ldots, a_k\}$, $I = \{b_1, \ldots, b_k\}$, where, assuming $J \geq I$, we have $a_k \geq b_k$, and $a_k \geq 1$, $b_k \geq 0$. Consider $J' = J - I_k = \{a_1 - 2^{k-1}, a_2 - 2^{k-2}, \ldots, a_k - 1\} \in \mathcal{R}$. Then $\xi^{\gamma(J)} = \xi^{\gamma(J')}\xi_k$ since $\gamma(J') = \gamma(J)$ except in the k-th place.

We have

$$\begin{aligned}
\langle \xi^{\gamma(J)}, \mathrm{Sq}^I \rangle &= \langle \xi^{\gamma(J')}\xi_k, \mathrm{Sq}^I \rangle \\
&= \langle \psi^*(\xi^{\gamma(J')} \otimes \xi_k), \mathrm{Sq}^I \rangle \\
&= \langle \xi^{\gamma(J')} \otimes \xi_k, \psi(\mathrm{Sq}^I) \rangle \\
&= \langle \xi^{\gamma(J')} \otimes \xi_k, \textstyle\sum_{I_1+I_2=I} \mathrm{Sq}^{I_1} \otimes \mathrm{Sq}^{I_2} \rangle \\
&= \textstyle\sum_{I_1+I_2=J} \langle \xi^{\gamma(J')}, \mathrm{Sq}^{I_1} \rangle \langle \xi_k, \mathrm{Sq}^{I_2} \rangle.
\end{aligned}$$

If $b_k = 0$, then all I_2 have 0 in the k-th place, so that $I_2 \neq I_k$ and $\langle \xi_k, \mathrm{Sq}^{I_2} \rangle = 0$.

If $b_k \neq 0$, the only non-zero summand occurs at $I_2 = I_k$. Hence $\langle \xi^{\gamma(J)}, \mathrm{Sq}^I \rangle = \langle \xi^{\gamma(J')}, \mathrm{Sq}^{I - I_k} \rangle$. The k-th position of $I - I_k$ is then one less than that of I_k, and likewise for that of J' and J. Descent on b_k and k completes the proof. □

Corollary 3.24. *As an algebra, $\mathcal{A}^*$ is the polynomial ring over $\mathbb{Z}_2$ generated by the $\{\xi_i\}$ $i \geq 1$.*

Proof. The above theorem shows that $\xi^{\gamma(J)}$, J admissible forms a $\mathbb{Z}_2$-basis for $\mathcal{A}^*$. The multiplication of $\mathcal{A}^*$ is commutative, and a polynomial ring is characterised by the fact that the monomials in the generators form a vector space basis. □

Example 3.25. Basis of $\mathcal{A}^*$ up to degree 7:

0	ξ_0			
1	ξ_1			
2	ξ_1^2			
3	ξ_1^3	ξ_2		
4	ξ_1^4	$\xi_1\xi_2$		
5	ξ_1^5	$\xi_1^2\xi_2$		
6	ξ_1^6	ξ_2^2	$\xi_2\xi_1^3$	
7	ξ_1^7	$\xi_2^2\xi_1$	$\xi_2\xi_1^4$	ξ_3

3.2.3 The Milnor basis

Since $\{\xi^R\}_{R\in\mathcal{R}}$ is a basis of $\mathcal{A}^*$, we can take the dual of these elements to get a basis for $\mathcal{A}$, known as the Milnor basis.

Definition 3.26 (Milnor basis)**.** The Milnor basis of $\mathcal{A}$ consists of the elements $\{\mathrm{Sq}^{(R)}\}_{R\in\mathcal{R}}$ such that $\begin{cases} \langle \xi^R, \mathrm{Sq}^{(R')}\rangle = 1 & \text{if } R = R', \\ \langle \xi^R, \mathrm{Sq}^{(R')}\rangle = 0 & \text{otherwise.} \end{cases}$

Remark 3.27. We use the notation $\mathrm{Sq}^{(R)}$ to distinguish from the usual notation for $\mathrm{Sq}^I = \mathrm{Sq}^{i_1}\mathrm{Sq}^{i_2}\cdots$ for $I = \{i_1, i_2, \ldots\}$.

Proposition 3.28. *For $I = \{i, 0, 0\ldots\}$, $\mathrm{Sq}^{(I)} = \mathrm{Sq}^i$.*

Proof. By definition, $\mathrm{Sq}^{(I)}$ is dual to $\xi_1^i \in \mathcal{A}^*$. Then $\xi^{\gamma(I)} = \xi^I = \xi_1^i$, and by theorem 3.23 we have $\langle \xi_1^i, \mathrm{Sq}^i\rangle = 1$. Conversely, if J is any other sequence of degree i, we have that $J > I$ (it will have non-zero entries further right), therefore again by theorem 3.23 we get $\langle \xi^{\gamma(J)}, \mathrm{Sq}^i\rangle = 0$. It follows that ξ_1^i is dual to Sq^i and so $\mathrm{Sq}^{(I)} = \mathrm{Sq}^i$. ☐

In the next section we will show that the co-product of $\mathcal{A}^*$ is given as

$$\varphi^*(\xi_k) = \sum_{i=0}^{k} (\xi_{k-i})^{2^i} \otimes \xi_i$$

We will assume this for now, and use it in the following example.

Example 3.29. Recall the subalgebra in example 3.8 in terms of the Serre-Cartan basis:

$$\begin{tikzcd}
 & \mathrm{Sq}^1 \arrow[r, dash] & \mathrm{Sq}^{\{2,1\}} \arrow[dr, dash] & & & \\
\mathrm{Sq}^0 \arrow[ur, dash] \arrow[r, dash] & \mathrm{Sq}^2 \arrow[rr, dash] \arrow[dr, dash] & & \mathrm{Sq}^{\{3,1\}} \arrow[rr, dash] & & \mathrm{Sq}^{\{5,1\}} \\
 & & \mathrm{Sq}^3 \arrow[rr, dash] & & \mathrm{Sq}^{\{4,1\}} + \mathrm{Sq}^5 \arrow[ur, dash] &
\end{tikzcd}$$

Then in terms of the Milnor basis we get:

$$\begin{tikzcd}
 & \mathrm{Sq}^{(1)} \arrow[r, dash] & \mathrm{Sq}^{(0,1)} \arrow[dr, dash] & & & \\
\mathrm{Sq}^{(0)} \arrow[ur, dash] \arrow[r, dash] & \mathrm{Sq}^{(2)} \arrow[rr, dash] \arrow[dr, dash] & & \mathrm{Sq}^{(1,1)} \arrow[rr, dash] & & \mathrm{Sq}^{(3,1)} \\
 & & \mathrm{Sq}^{(3)} \arrow[rr, dash] & & \mathrm{Sq}^{(2,1)} \arrow[ur, dash] &
\end{tikzcd}$$

- We get $\mathrm{Sq}^0 = \mathrm{Sq}^{(0)}$, $\mathrm{Sq}^1 = \mathrm{Sq}^{(1)}$, $\mathrm{Sq}^2 = \mathrm{Sq}^{(2)}$, $\mathrm{Sq}^3 = \mathrm{Sq}^{(3)}$ from the above position.

- By definition of ξ_2, $\langle \xi_2, \mathrm{Sq}^2 \mathrm{Sq}^1 \rangle = 1$. We also need to consider $\langle \xi_1^2, \mathrm{Sq}^2 \mathrm{Sq}^1 \rangle$. We have $\langle \xi_1^2, \varphi(\mathrm{Sq}^2 \otimes \mathrm{Sq}^1) \rangle = \langle \varphi^*(\xi_1^2), \mathrm{Sq}^2 \otimes \mathrm{Sq}^1 \rangle = \langle \varphi^*(\xi_1)^2, \mathrm{Sq}^2 \otimes \mathrm{Sq}^1 \rangle$. Then $\varphi^*(\xi_1)^2 = \xi_1^2 \otimes \xi_0 + \xi_0 \otimes \xi_1^2$ so that $\langle \xi_1^2, \mathrm{Sq}^2 \mathrm{Sq}^1 \rangle = \langle \xi_1^2, \mathrm{Sq}^2 \rangle \cdot \langle \xi_0, \mathrm{Sq}^1 \rangle + \langle \xi_0, \mathrm{Sq}^2 \rangle \cdot \langle \xi_1^2, \mathrm{Sq}^1 \rangle = 0$. Therefore $\mathrm{Sq}^2 \mathrm{Sq}^1 = \mathrm{Sq}^{(0,1)}$ as $\xi_2 = \xi^{\{0,1\}}$.
- By theorem 3.23, $\langle \xi^{\gamma(\{3,1\})}, \mathrm{Sq}^3 \mathrm{Sq}^1 \rangle = 1$, and $\gamma(\{3,1\}) = \{1,1\}$, $\xi^{\{1,1\}} = \xi_1 \xi_2$. To check $\langle \xi_1^4, \mathrm{Sq}^3 \mathrm{Sq}^1 \rangle$, we do the same as above and obtain $\langle \xi_1^4, \mathrm{Sq}^3 \mathrm{Sq}^1 \rangle = \langle \varphi^*(\xi_1)^4, \mathrm{Sq}^3 \otimes \mathrm{Sq}^1 \rangle$. Then $\varphi^*(\xi_1)^4 = \xi_1^4 \otimes \xi_0 + \xi_0 \otimes \xi_1^4$ and $\langle \xi_1^4, \mathrm{Sq}^3 \rangle \cdot \langle \xi_0, \mathrm{Sq}^1 \rangle + \langle \xi_0, \mathrm{Sq}^3 \rangle \cdot \langle \xi_1^4, \mathrm{Sq}^1 \rangle = 0$. Thus $\mathrm{Sq}^3 \mathrm{Sq}^1$ is $\mathrm{Sq}^{(1,1)}$ in the Milnor basis.
- We know $\mathrm{Sq}^{(5)} = \mathrm{Sq}^5$. Then $\gamma(\{4,1\}) = \{2,1\}$, so $\langle \xi_1^2 \xi_2, \mathrm{Sq}^4 \mathrm{Sq}^1 \rangle = 1$. But $\langle \xi_1^5, \mathrm{Sq}^4 \mathrm{Sq}^1 \rangle = 1$ also. We have $\varphi^*(\xi_1)^5 = \varphi^*(\xi_1)\varphi^*(\xi_1)^4 = \xi_1^5 \otimes \xi_0 + \xi_1^4 \otimes \xi_1 + \xi_1 \otimes \xi_1^4 + \xi_0 \otimes \xi_1^5$. Therefore $\langle \varphi^*(\xi_1)^5, \mathrm{Sq}^4 \otimes \mathrm{Sq}^1 \rangle = \langle \xi_1^5, \mathrm{Sq}^4 \rangle \langle \xi_0, \mathrm{Sq}^1 \rangle + \langle \xi_1^4, \mathrm{Sq}^4 \rangle \langle \xi_1, \mathrm{Sq}^1 \rangle + \langle \xi_1, \mathrm{Sq}^4 \rangle \langle \xi_1^4, \mathrm{Sq}^1 \rangle + \langle \xi_0, \mathrm{Sq}^4 \rangle \langle \xi_1^5, \mathrm{Sq}^1 \rangle$. The second term is 1, and all other zero. Thus in the Milnor basis, $\mathrm{Sq}^4 \mathrm{Sq}^1 = \mathrm{Sq}^{(2,1)} + \mathrm{Sq}^{(5)}$. Therefore $\mathrm{Sq}^5 + \mathrm{Sq}^{\{4,1\}} = \mathrm{Sq}^{(2,1)} + \mathrm{Sq}^{(5)} + \mathrm{Sq}^{(5)} = \mathrm{Sq}^{(2,1)}$.
- We have $\gamma(\{5,1\}) = \{3,1\}$ so $\langle \xi_1^3 \xi_2, \mathrm{Sq}^5 \mathrm{Sq}^1 \rangle = 1$. One can show as above that $\langle \xi_1^6, \mathrm{Sq}^5 \mathrm{Sq}^1 \rangle = 0$ and $\langle \xi_2^2, \mathrm{Sq}^5 \mathrm{Sq}^1 \rangle = 0$, so that $\mathrm{Sq}^5 \mathrm{Sq}^1 = \mathrm{Sq}^{(3,1)}$.

3.2.4 The diagonal map of $\mathcal{A}^*$

The work of this section is carried out by Milnor in [5, Chapter 4].

Since the multiplication of Steenrod squares is so complicated, we expect likewise that φ^* the co-product of $\mathcal{A}^*$ will be more difficult to describe than its multiplication. We were able to understand the structure of $\mathcal{A}$ by understanding $H^*(X)$ as an $\mathcal{A}$-module. Here we do something similar to understand $\mathcal{A}^*$.

Let $H_*(X) = H_*(X; \mathbb{Z}_2)$ Assume X is a complex of finite type (that is, the homology groups are finitely generated in each degree). Then $(H_*(X))^* = H^*(X)$. We have $H^*(X)$ an $\mathcal{A}$-module with $\mu: \mathcal{A} \otimes H^*(X) \to H^*(X)$.

The action of $\mathcal{A} \otimes \mathcal{A}$ on $H^*(X) \otimes H^*(X)$ that we defined earlier can be rewritten as

$$\mathcal{A} \otimes \mathcal{A} \otimes H^*(X) \otimes H^*(X) \xrightarrow{1 \otimes T \otimes 1} \mathcal{A} \otimes H^*(X) \otimes \ \mathcal{A} \otimes H^*(X) \xrightarrow{\mu \otimes \mu} H^*(X) \otimes H^*(X),$$

where T is the twist homomorphism permuting the factors. Let this be the second map accross the top row of the following diagram.

$\Delta: X \to X \times X$ induces $\Delta_*: H_*(X \times X) \to H_*(X) \otimes H_*(X)$ and $\Delta^*: H^*(X) \otimes H^*(X) \to H^*(X \times X)$, and we have that the following commutes:

$$\begin{tikzcd}
\mathcal{A} \otimes H^*(X) \otimes H^*(X) \arrow[r, "\psi"] \arrow[d, "1 \otimes \Delta^*"] & \mathcal{A} \otimes \mathcal{A} \otimes H^*(X) \otimes H^*(X) \arrow[r] & H^*(X) \otimes H^*(X) \arrow[d, "\Delta^*"] \\
\mathcal{A} \otimes H^*(X \times X) \arrow[rr, "\mu"] & & H^*(X \times X)
\end{tikzcd}$$

Let $\mu' = (\mu \otimes \mu)(1 \otimes T \otimes T)\psi$, then this μ' makes $H^*(X) \otimes H^*(X)$ an $\mathcal{A}$-module, and $H^*(X)$ is an algebra over $\mathcal{A}$.

We define a right-action of $\mathcal{A}$ on $H_*(X)$ by $\lambda: H_*(X) \otimes \mathcal{A} \to H_*(X)$ such that for $x \in H_*(X)$ and $\theta \in \mathcal{A}$, $\lambda(x, \theta)$ is the dual of y with $y \in H^*(X)$ such that x is dual to $\mu(\theta, y)$. So we get

$$\begin{array}{ccc} x & \mapsto & \lambda(x, \theta) \\ \mu(\theta, y) & \leftarrow\!\shortmid & y \end{array}$$

or that $\langle \lambda(x, \theta), y \rangle = \langle x, \mu(\theta, y) \rangle$. Dualizing, we have $\lambda^*: H^*(X) \to H^*(X) \otimes \mathcal{A}^*$.

Since $\lambda: H_*(X) \otimes \mathcal{A} \to H_*(X)$ is a right-action, the following commutes:

$$\begin{tikzcd}
H_*(X) \otimes \mathcal{A} \otimes \mathcal{A} \arrow[r, "1 \otimes \varphi"] \arrow[d, "\lambda \otimes 1"] & H_*(X) \otimes \mathcal{A} \arrow[d, "\lambda"] \\
H_*(X) \otimes \mathcal{A} \arrow[r, "\lambda"] & H_*(X)
\end{tikzcd}$$

Therefore, the dual diagram does also

$$\begin{tikzcd}
H^*(X)\otimes\mathcal{A}^*\otimes\mathcal{A}^* & H^*(X)\otimes\mathcal{A}^* \arrow[l, "1\otimes\varphi^*"'] \\
H^*(X)\otimes\mathcal{A}^* \arrow[u, "\lambda^*\otimes 1"] & H^*(X) \arrow[l, "\lambda^*"] \arrow[u, "\lambda^*"']
\end{tikzcd} \tag{3}$$

Better, we have

Proposition 3.30 ([5, Lemma 3]). *λ^* is an algebra homomorphism*

Sketch proof. One shows that the following diagram commutes:

$$\begin{tikzcd}
H_*(X)\otimes H_*(X)\otimes\mathcal{A}\otimes\mathcal{A} \arrow[r, "1\otimes T\otimes 1"] & H_*(X)\otimes\mathcal{A}\otimes H_*(X)\otimes\mathcal{A} \arrow[r, "\lambda\otimes\lambda"] & H_*(X)\otimes H_*(X) \\
H_*(X\times X)\otimes\mathcal{A} \arrow[u, "\Delta_*\otimes\psi"] \arrow[rr, "\lambda"] & & H_*(X\times X) \arrow[u, "\Delta_*"']
\end{tikzcd}$$

so that its dual commutes also.

Then letting $\alpha\otimes\beta\in H^*(X)\otimes H^*(X)$, and also using the isomorphism $\alpha\colon H^*(X)\otimes H^*(X)\to H^*(X\times X)$ given by the Künneth theroem, we can obtain $\lambda^*(\alpha\smile\beta)=\lambda^*(\alpha)\lambda^*(\beta)$. □

Next we will do something we've seen before: prove a formula by showing it checks out for the generator of $H^1(\mathbb{R}P^\infty)$. The following relation between λ^* and μ will help us.

Proposition 3.31. *For $y\in H^*(X)$, the following formulae are equivalent:*

1. *$\lambda^*(y)=\sum y_i\otimes w_i$ with $y_i\in H^*(X)$, ω_i in $\mathcal{A}^*$,*
2. *$\mu(\theta,y)=\sum\langle\theta,w_i\rangle y_i$ for all $\theta\in\mathcal{A}$.*

Proof. Assume 1. For $x\in H_*(X)$:

$$\begin{aligned}
\langle x,\mu(\theta,y)\rangle &= \langle\lambda(x,\theta),y\rangle\\
&= \langle x\otimes\theta,\lambda^*(y)\rangle\\
&= \textstyle\sum\langle x,y_i\rangle\langle\theta,w_i\rangle\\
&= \langle x,\textstyle\sum\langle\theta,w_i\rangle y_i\rangle
\end{aligned}$$

Conversely, assume 2. Then for $\theta\in\mathcal{A}$,

$$\begin{aligned}
\langle x\otimes\theta,\lambda^*y\rangle &= \langle\lambda(x,\theta),y\rangle\\
&= \langle x,\mu(\theta,y)\rangle\\
&= \langle x,\textstyle\sum\langle\theta,w_i\rangle y_i\rangle\\
&= \textstyle\sum\langle x,y_i\rangle\langle\theta,w_i\rangle\\
&= \langle x\otimes\theta,\textstyle\sum y_i\otimes w_i\rangle.
\end{aligned}$$

□

Proposition 3.32. *Let $\alpha\in H^1(\mathbb{R}P^\infty)$ be the generator. Then $\lambda^*(\alpha)=\sum_{i\geq 0}\alpha^{2^i}\otimes\xi_i$.*

Proof. Equivalently, we'll show $\mu(\theta,\alpha)=\sum_{i\geq 0}\langle\theta,\xi_i\rangle\cdot\alpha^{2^i}$ for all $\theta\in\mathcal{A}$. It is enough to check this for Sq^I with I admissible. $\langle\mathrm{Sq}^I,\xi_i\rangle=1$ when $I=I_i$ and 0 otherwise. Therefore $\sum_{i\geq 0}\langle\xi_i,\mathrm{Sq}^I\rangle\cdot\alpha^{2^i}=0$ unless $I=I_i$, where we get α^{2^i}. But then $\mu(\mathrm{Sq}^{I_i},\alpha)=\mathrm{Sq}^{I^i}(\alpha)=\alpha^{2^i}$. □

Corollary 3.33. $\lambda^*(\alpha^{2^i})=\sum_{j\geq 0}\alpha^{2^{i+j}}\otimes(\xi_j)^{2^i}$.

Proof. Since λ^* is an algebra homomorphism, we get $\lambda^*(\alpha^{2^i})=(\lambda^*\alpha)^{2^i}=(\sum\alpha^{2^j}\otimes\xi_j)^{2^i}=\sum_j(x^{2^j}\otimes\xi_j)^{2^i}=\sum_j x^{2^{i+j}}\otimes(\xi_j)^{2^i}$. □

Theorem 3.34. *The coproduct $\varphi^*\colon \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ of $\mathcal{A}^*$ is given by the formula*

$$\varphi^*(\xi_k) = \sum_{i=0}^{k} (\xi_{k-i})^{2^i} \otimes \xi_i$$

Proof. Let $\alpha \in H^1(\mathbb{R}P^\infty)$. The following two expressions are equal from (3):

$$(1 \otimes \varphi^*)\lambda^*(\alpha) = (1 \otimes \varphi^*)(\sum_{k \geq 0} \alpha^{2^k} \otimes \xi_k) = \sum_{k \geq 0} \alpha^{2^k} \otimes \varphi^*(\xi_k),$$

and

$$(\lambda^* \otimes 1)\lambda^*(\alpha) = (\lambda^* \otimes 1)(\sum_{i \geq 0} \alpha^{2^i} \otimes \xi_i) = \sum_{i \geq 0} \sum_{j \geq 0} \alpha^{2^{i+j}} \otimes (\xi_j)^{2^i} \otimes \xi_i.$$

Therefore $\varphi^*(\xi_k) = \sum_{i+j=k} (\xi_j)^{2^i} \otimes \xi_i$, as required. □

We do not explore applications of the Milnor basis or the coproduct in this report, but give one example where propositon 3.28 allows us to describe the primitives of $\mathcal{A}^*$.

Definition 3.35 (Primitive element)**.** For A a co-algebra with unit 1 and co-product $\psi\colon A \to A \otimes A$, we say that $a \in A$ is primitive if $\psi(a) = a \otimes 1 + 1 \otimes a$ for 1 the unit of A.

Example 3.36. Sq^4 is dual to ξ_1^4, and $\varphi^*(\xi_1^4) = (\varphi^*(\xi_1))^4 = (\sum_{i=0}^{1} \xi_{1-i}^{2^i} \otimes \xi_i)^4 = (\xi_1 \otimes \xi_0 + \xi_0^2 \otimes \xi_1)^4 = (\xi_1 \otimes \xi_0)^4 + (\xi_0 \otimes \xi_1)^4 = \xi_1^4 \otimes \xi_0 + \xi_0 \otimes \xi_1^4$, and so ξ_1^4 is primitive in $\mathcal{A}^*$, since ξ_0 is the unit of $\mathcal{A}^*$.

Recall that the indecomposables of $\mathcal{A}$ are $\{\mathrm{Sq}^{2^k}\}_{k \geq 0}$ as shown in theorem 3.10. The dual of these in $\mathcal{A}^*$ are $\{\xi_1^{2^k}\}_{k \geq 0}$ by proposition 3.28. We admit the following:

Proposition 3.37. *There is a bijection between the indecomposables of $\mathcal{A}$ and the primitives of $\mathcal{A}^*$ by sending $\theta \in \mathcal{A}$ to its dual in $\mathcal{A}^*$.*

That the indecomposables get mapped to primitives is the same argument as the above example with 4 replaced by 2^k. Thus we get

Corollary 3.38. *The primitive elements of $\mathcal{A}^*$ are $\{\xi_1^{2^k}\}_{k \geq 0}$.*

4 Applications

4.1 Cohomology of Eilenberg-MacLane spaces

We expect the Steenrod squares to play a role in the description of the cohomology of the Eilenberg-MacLane spaces $K(\mathbb{Z}_2, n)$. Here we make that expectation explicit. We use a combination of path space fibrations over $K(\mathbb{Z}_2, n)$, the Serre spectral sequence, and the Steenrod squares to obtain a description of $H^*(K(\mathbb{Z}_2, n))$.

We will not explain the Serre spectral sequence in detail here. A reference is chapters 5,6 of [6].

4.1.1 Serre spectral sequence for fibrations

Fibrations induce long exact sequences of homotopy groups, but the same is not true of cohomology groups. Instead we can relate the cohomology of the spaces of a fibration in a more complicated manner - namely by the use of Serre's spectral sequence.

Whenever we refer to a fibration, we will mean a Serre fibration, which is defined as follows:

Definition 4.1 (Serre fibration)**.** A map $p: E \to B$ is a Serre fibration if it satisfies the homotopy lifting property for CW complexes. That is, for X a CW complex and a homotopy $H: X \times I \to B$ with a lift $\tilde{H}_0$ of H_0 (i.e. $p\tilde{H}_0 = H_0$), we can lift H to a homotopy $\tilde{H}: X \times I \to E$ such that $p\tilde{H} = H$.

$$\begin{tikzcd}
X \times \{0\} \arrow[r, "\tilde{H}_0"] \arrow[d] & E \arrow[d, "p"] \\
X \times I \arrow[r, "H"] \arrow[ur, dashed, "\tilde{H}"] & B
\end{tikzcd}$$

We call B the base, E the total space. If B is path-connected, then all $p^{-1}(b)$ for all $b \in B$ are homotopic, and we let $p^{-1}(b) \simeq F$ be the fiber.

Proposition 4.2 (LES of homotopy groups - [1, Theorem 4.41])**.** *Suppose $p: E \to B$ is a Serre fibration. Choose basepoints $b_0 \in B$ and $p^{-1}(b_0) = x_0 \in F$. Then the map $p_*: \pi_n(E, F, x_0) \to \pi_n(B, b_0)$ is an isomorphism for all $n \geq 1$. If B is path-connected, then there is a long exact sequence*

$$\cdots \pi_n(F, x_0) \to \pi_n(E, x_0) \xrightarrow{p_*} \pi_n(B, b_0) \to \pi_{n-1}(F, x_0) \to \cdots \to \pi_0(E, x_0) \to 0$$

Example 4.3 (Path space fibrations)**.** Let (X, x_0) be a pointed space, PX the path space of X, i.e. the space of pointed maps $\mathrm{map}_*(I, X)$, and ΩX the loop-space of X, i.e. the space of pointed maps $\mathrm{map}_*(S^1, X)$. Then the map $PX \to X$ defined by evaluating a path at 1 is a fibration, with fiber ΩX. (see [1] p407-408.)

By truncating paths, one sees that PX is contractible. We will denote these path space fibrations as $\Omega X \to * \to X$, where $*$ denotes a contractible space.

Example 4.4 ([7, Chapter 9, p84])**.** Consider a short exact sequence of abelian groups $0 \to A \to B \to C \to 0$. There is an associated fibration $K(A, n) \to K(B, n) \to K(C, n)$ constructed by realizing the homomorphism $B \to C$ by a map $K(B, n) \to K(C, n)$ as in example 1.25, and then converting it into a fibration. The long exact sequence of homotopy groups shows that the fibre is a $K(A, n)$.

The Serre spectral sequence relates the cohomology of the spaces in a Serre fibration:

Theorem 4.5 (Serre spectral sequence - cohomological version)**.** *Let R be a commutative ring. Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fibration with F, B path-connected and $\pi_1(B)$ acting trivially on $H^*(F; R)$.*

Then there is a first quadrant cohomological spectral sequence of algebras $\{E_r^{*,*}, d_r\}$ converging to $H^*(E;R)$ as an algebra, with E_2 page

$$E_2^{p,q} = H^p(B; H^q(F,R)) \Rightarrow H^{p+q}(E;R)$$

The stable terms $E^{p,n-p}$ are isomorphic to the quotients F_p^n/F_{p+1}^n in a decreasing filtration $0 \subset F_n^n \subset \cdots \subset F_0^n = H^n(E;R)$ of $H^n(E;R)$.

Remark 4.6. In nearly all cases we deal with, B is simply-connected, so we don't need to consider the action of $\pi_1(B)$.

Remark 4.7. By the univesal coefficient theorem, $H^n(X;G) \simeq H^n(X;\mathbb{Z}) \otimes G \oplus \mathrm{Tor}(H^{n+1}(X;\mathbb{Z});G)$. Therefore if k is a field, $E_2^{*,*} = H^*(B;k) \otimes H^*(F;k)$, since the Tor terms vanish. We'll be working with $k = \mathbb{Z}_2$.

Let $F \to E \xrightarrow{p} B$ be a Serre fibration. Let $p_0\colon (E,F) \to (B,b_0)$ be the induced fibration of pairs, where b_0 is the basepoint of B. We have a LES of cohomology for the pair (B,b_0), as well as for the pair (E,F). Then the following diagram commutes:

$$\begin{tikzcd}
{} \arrow[r] & H^{n-1}(b_0) \arrow[r, "\delta"] \arrow[d] & H^n(B,b_0) \arrow[r, "j^*"] \arrow[d, "p_0^*"] & H^n(B) \arrow[r] \arrow[d, "p^*"] & H^n(b_0) \arrow[r, "\delta"] \arrow[d] & {} \\
{} \arrow[r] & H^{n-1}(F) \arrow[r, "\delta"] \arrow[urr, dashed] & H^n(E,F) \arrow[r, "j^*"] & H^n(E) \arrow[r] & H^n(F) \arrow[r, "\delta"] & {}
\end{tikzcd}$$

We define the transgression homomorphism $\tau\colon \delta^{-1}(\mathrm{Im}\, p_0^*) \to H^n(B)/j^*(\ker p_0^*)$ by $\tau(z) = j^*(r) + j^*(\ker p_0^*)$ where $z \in \delta^{-1}(\mathrm{Im}\, p_0^*)$ and $p_0^*(r + \ker p_0^*) = \delta(z)$. This relates to the Serre spectral sequence as follows:

Proposition 4.8 ([6, Theorem 6.8])**.** *Given a fibration $F \to E \xrightarrow{p} B$ with B, F path-connected, then for the associated Serre spectral sequence we have*

- $E_n^{0,n-1} \simeq \delta^{-1}(\mathrm{Im}\, p_0^*) \subset H^{n-1}(F)$,
- $E_n^{n,0} \simeq H^n(B)/\ker p^*$,
- $d_n\colon E_n^{0,n-1} \to E_n^{n,0}$ *is the transgression* τ.

Therefore we call the differentials $d_n\colon E_n^{0,n-1} \to E_n^{n,0}$ transgressions. We will often denote these by τ. We say that a class $x \in H^*(F)$ is transgressive, or that it transgresses if $x \in \bigcap_{i=1}^{n-1} \ker d_i$, i.e. that it persists to the E_n page so that $d_n(x)$ is defined. We will just need to consider transgressions in this sense, but we needed to introduce the former definition to state the following:

Proposition 4.9. *If x is transgressive, then $\mathrm{Sq}^i x$ is transgressive, and $\mathrm{Sq}^i(\tau x) = \tau(\mathrm{Sq}^i x)$*

Proof. This follows from the naturality of the squares, and that they commute with coboundary maps for pairs, by lemma 2.5. □

4.1.2 Serre's theorem

Recall that $e(I)$ is the excess of a sequence. We will prove the following result:

Theorem 4.10 (Serre's theorem)**.**

1. *$H^*(K(\mathbb{Z}_2,n);\mathbb{Z}_2)$ for $n \geq 1$ is the polynomial ring on generators $\mathrm{Sq}^I \iota_n$, where ι_n is the fundamental class of $K(\mathbb{Z}_2,n)$ and I ranges over all admissible sequences with $e(I) < n$.*
2. *$H^*(K(\mathbb{Z},n);\mathbb{Z}_2)$ for $n \geq 2$ is the polynomial ring on the generators $\mathrm{Sq}^I \iota_n$, where ι_n is the reduction mod 2 of the fundamental class of $K(\mathbb{Z},n)$, and I ranges over all admissible sequences with $e(I) < n$ and having no Sq^1 term.*

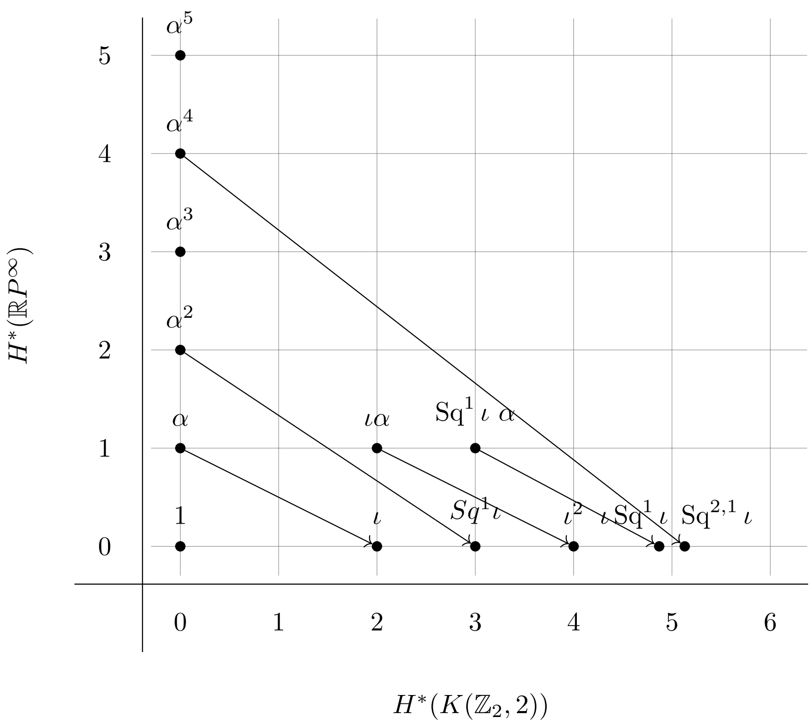
3. *$H^*(K(\mathbb{Z}_{2^k}, n); \mathbb{Z}_2)$ for $k > 1$ and $n \geq 2$ is the polynomial ring on generators $\operatorname{Sq}^I \iota_n$ and $\operatorname{Sq}^I \kappa_{n+1}$ for ι_n the reduction mod 2 of the fundamental class of $K(\mathbb{Z}_{2^k}, n)$, and κ_{n+1} a generator of $H^{n+1}(K(\mathbb{Z}_{2^k}, n); \mathbb{Z}_2)$. The sequences I range over all admissible sequences having no Sq^1 term, with $e(I) < n$ for $\operatorname{Sq}^I \iota_n$ and $e(I) \leq n$ for $\operatorname{Sq}^I \kappa_{n+1}$.*

The proof of Serre's theorem is an inductive argument using path space fibrations over $K(G, n)$, so we detail the case $K(G, 1) \to * \to K(G, 2)$. The following examples compute the cohomology with $\mathbb{Z}_2$ coefficients of the second Eilenberg-MacLane space for $G = \mathbb{Z}_2$, $\mathbb{Z}$, $\mathbb{Z}_{2^k}$, making use of proposition 4.9.

Example 4.11 (Cohomology of $K(\mathbb{Z}_2, 2)$)**.** Consider the path space fibration $\mathbb{R}P^\infty = K(\mathbb{Z}_2, 1) \to * \to K(\mathbb{Z}_2, 2)$. Then $H^*(\mathbb{R}P^\infty) \simeq \mathbb{Z}_2[\alpha]$ for $\alpha \in H^1(\mathbb{R}P^\infty)$. Note that $\alpha^{2^k} = \operatorname{Sq}^{I_k}(\alpha)$.

The zeroth column of the E_2 page is the cohomology of $\mathbb{R}P^\infty$, and the first column is zero since $K(\mathbb{Z}_2, 2)$ is 1-connected. The spectral sequence converges to a contractible space, so that the E_∞ page has $\mathbb{Z}_2$ in the $(0, 0)$ spot and is trivial elsewhere. We use this to understand the cohomology of $K(\mathbb{Z}_2, 2)$, which we will only describe here loosely.

For brevity, we will include some higher differentials on the E_2 page, and the black dots denote copies of $\mathbb{Z}_2$. Note that we are not including all differentials (not even all d_2 differentials), or elements of the E_2 page on the diagram. We will include enough to describe the first few cohomology groups of $K(\mathbb{Z}_2, 2)$.



The only possible non-trivial differential on α is the d_2 differential. We cannot have $d_2(\alpha) = 0$, else α would persist to the E_∞ page, so let $d_2(\alpha) = \iota \in H^2(K(\mathbb{Z}_2, 2))$. This spot in the spectral sequence can only be hit by a d_2 differential, so to "kill" this spot we must have $d_2\colon E_2^{0,1} \to E_2^{2,0}$ an isomorphism. Since α transgresses to ι, by proposition 4.9 it follows that $\alpha^{2^k} = \operatorname{Sq}^{I_k} \alpha$ transgresses to $\operatorname{Sq}^{I_k} \iota$ for $I_k = \{2^{k-1}, \ldots, 2, 1\}$.

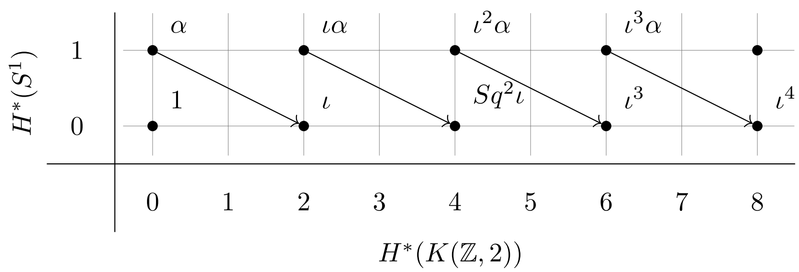
As d_2 is a derivation, $d_2(\iota^j \alpha^k) = k\iota^{j+1}\alpha^{k-1} \neq 0$ for k odd. We have $d_2(\alpha^3) = \iota\alpha^2$, so that α^3 does not transgress. Then $d_2(\iota\alpha) = \iota^2 = \operatorname{Sq}^2 \iota$, and the d_2 differential in this spot is an isomorphism. We also have $d_2(\operatorname{Sq}^1 \iota\ \alpha) = \iota \operatorname{Sq}^1 \iota$

By Serre's theorem 4.10, we expect the cohomology of $K(\mathbb{Z}_2, 2)$ to be polynomial in $\mathrm{Sq}^{I_k}\iota$ for $k \geq 0$ and $I_k = \{2^{k-1}, \dots, 2, 1\}$, since these are the admissible sequences of degree less than 2. We begin to see this emerge here.

The cohomology of $K(\mathbb{Z}_2, 2)$ up to degree 5 is thus as follows, writing generators:

k	0	1	2	3	4	5
$H^k(K(\mathbb{Z}_2, 2))$	1	0	ι	$\mathrm{Sq}^1\iota$	ι^2	$\iota\,\mathrm{Sq}^1$, $\mathrm{Sq}^{2,1}\iota$

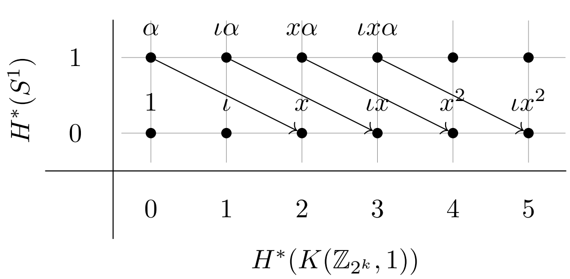
Example 4.12 (Cohomology of $K(\mathbb{Z}, 2)$)**.** We use the Serre SS associated with the fibration $S^1 = K(\mathbb{Z}, 1) \to * \to K(\mathbb{Z}, 2)$, with the black dots on the E_2 page denoting copies of $\mathbb{Z}_2$. The cohomology of S^1 is $\mathbb{Z}_2$ in degrees 0, 1, and zero elsewhere.



Since the spectral sequence converges to a contractible space, the class $\alpha \in H^1(S^1)$ must transgress to some class $\iota \in H^2(K(\mathbb{Z}_2, n))$, and moreover $d_2\colon H^1(S^1) \to H^2(K(\mathbb{Z}, 2))$ must also be surjective so that the cohomology of $H^2(K(\mathbb{Z}, 2))$ gets "killed off". Thus $d_2(\alpha) = \iota$ for ι a class that generates $H^2(K(\mathbb{Z}_2, 2))$. Then $d_2(\iota\alpha) = \iota^2 = \mathrm{Sq}^2\iota$ and more generally $d_2(\iota^n\alpha) = \iota^{n+1}$. Then $E_3 = E_\infty$ and we see that $H^*(K(\mathbb{Z}, 2)) = \mathbb{Z}_2[\iota]$ for ι a class of degree 2.

Example 4.13 (Cohomology of $K(\mathbb{Z}_{2^k}, 2)$)**.** Let $k > 1$ and consider the fibration $K(\mathbb{Z}, 1) \to K(\mathbb{Z}, 1) \to K(\mathbb{Z}_{2^k}, 1)$ induced by the short exact sequence $0 \to \mathbb{Z} \xrightarrow{2^k} \mathbb{Z} \to \mathbb{Z}_{2^k} \to 0$ (see example 4.4). For the associated Serre SS, the total space is S^1, so only the $(0,0)$-th spot and the first diagonal of the E_∞ page can be non-trivial.

As the base space is a $K(\mathbb{Z}_{2^k}, 1)$ we know that its first $\mathbb{Z}_2$-cohomology group is $\mathbb{Z}_{2^k} \otimes \mathbb{Z}_2 = \mathbb{Z}_2$, generated by the reduction modulo 2 of the fundamental class of $H^1(K(\mathbb{Z}_{2^k}; 1); \mathbb{Z}_{2^k})$, we'll call this ι. This class cannot be hit by a differential, so that it persists to the E_∞ page. The first diagonal should be a filtration of $H^1(S^1; \mathbb{Z}_2) = \mathbb{Z}_2$, meaning the $(0,1)$-th spot on the E_∞ page must be zero, and so $\alpha \in H^1(S^1)$ must transgress to some class in $H^2(K(\mathbb{Z}_{2^k}, n))$. The d_2 differential in this spot must be an isomorphism. Therefore $d_2(\alpha) = x \in H^2(K(\mathbb{Z}_{2^k}, 1))$ the generator of this cohomology group, and $d_2(x^n\alpha) = x^{n+1}$, $d_2(\iota x^n\alpha) = \iota x^{n+1}$.



We get $E_3 = E_\infty$ and $H^*(K(\mathbb{Z}_{2^k}, 2)) = \mathbb{Z}_2[x] \otimes \Lambda[\iota]$, where $\Lambda[\iota]$ is the exterior algebra on generator ι. Since ι is the reduction mod 2 of a class with coefficients in $\mathbb{Z}_{2^k}$, it follows that the Bockstein Sq^1 is zero on ι, so that $\iota^2 = \mathrm{Sq}^1\iota = 0$ (informally, we can think of $\beta = \mathrm{Sq}^1$ as $\mathrm{Sq}^1(\iota) = (\frac{1}{2}\delta\iota) \mod 2$ where δ is the coboundary).

Later, we will obtain a more explicit description of this cohomology class x.

Remark 4.14. In the last example the base space is not simply-connected. It can be shown that $\pi_1(B)$ acts trivially on $H^*(S^1)$ (since $H^0(S^1) = H^1(S^1) = \mathbb{Z}_2$).

Given a fibration $F \to E \to B$, we can understand some of the cohomology of the base space by understanding the cohomology of the fibre, and what classes transgress. In certain cases this information is enough to be able to describe the cohomology of the base completely, which is what Borel's theorem describes. We first need the following definition:

Definition 4.15 (Simple system of generators)**.** Let R be a graded commutative ring over a field k. A simple system of generators of R is an ordered set $\{x_1, x_2, \ldots\}$ such that $x_i \in R$ and the monomials $\{x_{i_1} x_{i_2} \cdots x_{i_r} | \ i_1 < \cdots < i_r\}$ form a k-basis for R.

Example 4.16.

1. $H^*(\mathbb{R}P^\infty) = \mathbb{Z}_2[\alpha]$ for $|\alpha| = 1$ has a simple system of generators $\{\alpha, \alpha^2, \alpha^4, \ldots, \alpha^{2^k} \ldots\} = \{\alpha, \mathrm{Sq}^1 \alpha, \mathrm{Sq}^{\{2,1\}} \alpha, \ldots, \mathrm{Sq}^{I_k} \alpha \ldots\}$,
2. In example 4.12, $H^*(K(\mathbb{Z}, 2)) = \mathbb{Z}_2[\iota]$ for ι a class of degree 2. This has a simple system of generators $\{\iota, \iota^2, \iota^4, \ldots\} = \{\iota, \mathrm{Sq}^2 \iota, \mathrm{Sq}^{\{4,2\}} \iota, \ldots\}$,
3. In example 4.13, $H^*(K(\mathbb{Z}_{2^k}, 2)) = \mathbb{Z}_2[x] \otimes \Lambda[\iota]$. This has a simple system of generators $\{\iota, x, x^2, x^4, \ldots\} = \{\iota, x, \mathrm{Sq}^2 x, \mathrm{Sq}^{\{4,2\}} x, \ldots\}$.

Theorem 4.17 (Borel's theorem, [8, Chapter 9, Theorem 1])**.** *Let k be a field. Let $F \to E \to B$ be a fibration with E contractible, and B simply-connected. If $H^*(F; k)$ has a simple system of generators $\{x_\alpha\}_\alpha$ with x_α transgressive (and these classes are of odd dimension if char $k \neq 2$), then $H^*(B; k)$ is the polynomial algebra on $\{\tau x_\alpha\}_\alpha$ where τ is the transgression.*

Remark 4.18. The proof of Borel's theorem uses Zeeman's comparison theorem for spectral sequences. We build a model for the spectral sequence that resembles the result of the theorem, then show that this model spectral sequence and our actual spectral sequence are similar enough so that we can apply the comparison theorem and say that the $E_2^{*,0}$ terms are isomorphic, giving us the desired cohomology of the base space.

Finally, before we give a proof of Serre's theorem, we justify the bound on the excess of the admissible sequences in the statement of the theorem.

Lemma 4.19 ([7, Lemma 5.33])**.** *Let ι_n be of degree n,*

1. *If $I = \{i_1, i_2, \ldots, i_r\}$ is admissible with $e(I) > n$, then $\mathrm{Sq}^I \iota_n = 0$,*
2. *The $\mathrm{Sq}^I \iota_n$ for I admissible with $e(I) = n$ are exactly of the form $(\mathrm{Sq}^J \iota_n)^{2^k}$ for J admissible with $e(J) < n$ and $k > 0$.*

Proof.

1. If $e(I) > n$, then $i_1 - i_2 - \cdots - i_r > n$. Therefore $i_1 > n + i_2 + \cdots + i_r = |\mathrm{Sq}^{i_2} \cdots \mathrm{Sq}^{i_r} \iota_n|$, so that $\mathrm{Sq}^{i_1}(\mathrm{Sq}^{i_2} \cdots \mathrm{Sq}^{i_r} \iota_n) = 0$ because of the degrees.
2. If $e(I) = n$, then $i_1 = n + i_2 + \cdots + i_r$, so $\mathrm{Sq}^{i_1}(\mathrm{Sq}^{i_2} \cdots \mathrm{Sq}^{i_r} \iota_n) = (\mathrm{Sq}^{i_2} \cdots \mathrm{Sq}^{i_r} \iota_n)^2$ and $e(\{i_2, \ldots i_r\}) \leq n$. If $e(\{i_2, \ldots i_r\}) < n$, we're done. Else, repeat, the process must terminate.

   Conversely, consider $\mathrm{Sq}^{i_2} \cdots \mathrm{Sq}^{i_r} \iota_n$ with $\{i_2, \ldots, i_r\}$ admissible and $e(\{i_2, \ldots, i_r\}) \leq n$. Then $(\mathrm{Sq}^{i_2} \cdots \mathrm{Sq}^{i_r} \iota_n)^2 = \mathrm{Sq}^{i_1} \mathrm{Sq}^{i_2} \cdots \mathrm{Sq}^{i_r} \iota_n$ for $i_1 = n + i_2 + \cdots + i_r$, and $I = \{i_1, \ldots, i_r\}$ is admissible since $i_1 - 2i_2 = n - i_2 + i_3 + \cdots + i_r = n - e(\{i_2, \ldots, i_r\}) \geq 0$. Moreover, $e(I) = n$. Therefore for any $\mathrm{Sq}^J(\iota_n)$ with J admissible and $e(J) < n$, we can take some 2^k-th power for $k > 0$ to obtain $(\mathrm{Sq}^J \iota_n)^{2^k} = \mathrm{Sq}^I \iota_n$ with I admissible and $e(I) = n$.

$\square$

Proof of Serre's theorem. We prove the case for $K(\mathbb{Z}_2, n)$ by induction on n.

For $n = 1$, we have the fibration $\mathbb{R}P^\infty = K(\mathbb{Z}_2, 1) \to * \to K(\mathbb{Z}_2, 2)$. Then $H^*(K(\mathbb{Z}_2, 1); \mathbb{Z}_2)$ has a simple system of transgressive generators as seen in example 4.11, and a contractible total space, so that we can apply Borel's theorem. We obtain that $H^*(K(\mathbb{Z}_2, 2); \mathbb{Z}_2)$ is the polynomial ring $\mathbb{Z}_2[\mathrm{Sq}^{I_k}]$ where $I^k = \{2^{k-1}, \ldots, 2, 1\}$ for $k \geq 0$, which are precisely the admissible sequences of excess less than 2.

For the general case, we proceed similarly. Consider the fibration $K(\mathbb{Z}_2, n) \to * \to K(\mathbb{Z}_2, n+1)$. Assume inductively that $H^*(K(\mathbb{Z}_2, n); \mathbb{Z}_2) \simeq \mathbb{Z}_2[\mathrm{Sq}^I \iota_n]$ for I admissible sequences with $e(I) < n$. The simple system of generators for this ring are $\{(\mathrm{Sq}^I \iota_n)^{2^i}\}_{i,I}$ ranging over index i and admissible sequences I. Since $K(\mathbb{Z}_2, n)$, $K(\mathbb{Z}_2, n+1)$ are $n-1$ and n-connected respectively, and the spectral sequence converges to the cohomology of a contractible space, we must have $\tau(\iota_n) = \iota_{n+1}$, so that ι_n is transgressive, and so $\{(\mathrm{Sq}^I \iota_n)^{2^i}\}_{i,I}$ are all transgressive by proposition 4.9. Note that $(\mathrm{Sq}^I \iota_n)^{2^i}$ is of the form $\mathrm{Sq}^J \iota_n$ with J of excess $\leq n$ by lemma 4.19. Therefore by Borel's theorem $H^*(K(\mathbb{Z}_2, n+1); \mathbb{Z}_2)$ is the polynomial ring $\mathbb{Z}_2[\mathrm{Sq}^J \iota_{n+1}]$ with excess $\leq$ n, i.e. $< n+1$.

The proofs of part 2, 3 of Serre's theorem are analogous, where we use examples 4.12 and 4.13 for the cohomology of the second Eilenberg-MacLane spaces for the base cases. □

We conclude by stating that the Steenrod algebra generates all stable cohomology operations between cohomology groups with $\mathbb{Z}_2$ coefficients.

Theorem 4.20 ([7, Corollary 5.38]). *$\mathcal{A} \to H^*(K(\mathbb{Z}_2, n); \mathbb{Z}_2)$, $\mathrm{Sq}^I \mapsto \mathrm{Sq}^I(\iota_n)$ is an isomorphism from the degree d part of $\mathcal{A}$ onto $H^{n+d}(K(\mathbb{Z}_2, n); \mathbb{Z}_2)$ for degrees $d \leq n$.*

Corollary 4.21. *$\mathcal{A} \simeq \varprojlim_n \tilde{H}^{n+*}(K(\mathbb{Z}_2, n); \mathbb{Z}_2)$ with the inverse limit as described in remark 1.29, so that the Steenrod algebra classifies stable cohomology operations.*

4.2 2-components of stable homotopy groups of spheres

The p-component of an abelian group A is the quotient group obtained by factoring out the subgroup of all torsion elements of order prime to p in A.

In this section we calculate the 2-components of the first few stable homotopy groups of spheres. We do this by working with cohomology in $\mathbb{Z}_2$ coefficients and building a tower of better approximations to S^n. In this section, by "cohomology groups" we will mean those with $\mathbb{Z}_2$ coefficients, and by "homotopy groups" we will mean the 2-components of the homotopy groups.

The end result will look as follows:

$$\begin{tikzcd}
K(\mathbb{Z}_8, n+3) \arrow[r] & X_3 \arrow[d] & \\
K(\mathbb{Z}_2, n+2) \arrow[r] & X_2 \arrow[d] \arrow[r, "p_2^* p_1^* \mathrm{Sq}^4 \iota_n"] & K(\mathbb{Z}_8, n+4) \\
K(\mathbb{Z}_2, n+1) \arrow[r] & X_1 \arrow[d] \arrow[r, "\alpha"] & K(\mathbb{Z}_2, n+3) \\
 & X_0 = K(\mathbb{Z}, n) \arrow[r, "\mathrm{Sq}^2"] & K(\mathbb{Z}_2, n+2)
\end{tikzcd} \tag{4}$$

and the 2-components of the first six stable homotopy groups of spheres being

k	0	1	2	3	4	5
$\pi_{n+k}(S^n)$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_8$	0	0

By "better approximations to S^n", we mean that the cohomology of the spaces X_i will give succesively better approximations to the cohomology of S^n. If we have a map from S^n to one of these spaces that induces an isomorphism of cohomology up to a certain degree, then this will induce an isomorphism of their homotopy groups up to a certain degree also. This is the statement of Theorem 4.30.

Fistly, the map $f\colon S^n \to K(\mathbb{Z}, n)$ is taken as the generator of $\pi_n(K(\mathbb{Z}, n))$. The cohomology of $K(\mathbb{Z}, n)$ is the same as that of S^n up to degree $n+1$, which is seen by Serre's theorem 4.10. In degree $n+2$, the cohomology of the spaces differs because of the generator $\mathrm{Sq}^2\,\iota_n$ of $H^{n+2}(K(\mathbb{Z}, n))$. We want to "kill off" this cohomology class to obtain a new space X_1 with trivial cohomology in degree $n+2$. To do this, we represent $\mathrm{Sq}^2\,\iota_n$ as a map $\mathrm{Sq}^2\colon K(\mathbb{Z}, n) \to K(\mathbb{Z}_2, n+2)$. Then, we pull back the path space fibration over $K(\mathbb{Z}_2, n+2)$ along f to obtain a fibration $K(\mathbb{Z}_2, n+1) \to X_1 \to K(\mathbb{Z}, n)$. For this to have been worthwhile, we need two things to be true:

1. There is a map $f_1\colon S^n \to X_1$,
2. The cohomology of X_1 is trivial in degree $n+2$.

To get f_1, we show in proposition 4.25 that we can lift f from a map to the base to a map to the total space because $\mathrm{Sq}^2 \circ f$ is null-homotopic.

For the second point, we use the Serre SS associated to the fibration to determine the cohomology of X_1. For the degree $n+1$ cohomology of X_1 to be trivial, we need that $\iota_{n+1} \in H^{n+1}(K(\mathbb{Z}_2, n+1))$ transgresses to $\mathrm{Sq}^2\,\iota_n \in H^{n+2}(K(\mathbb{Z}_2, n+2))$. This is verified in lemma 4.24. After this, all other transgression computations for this fibration are made easy by the squares commuting with the transgression, as stated in proposition 4.9. Moreover, looking at transgression computations is enough to determine the cohomology of X_1 because of the connected-ness of the base and fiber, as we will see in theorem 4.23.

Then we repeat the strategy: find the first non-trivial cohomology class of X_1 in degree greater than n, and kill this off by obtaining a new fibration with total space X_2. Once more we use the Serre SS to obtain the cohomology of X_2. This time, the cohomology of the base X_1 is not expressed in terms of the squares, so our transgression calculations become harder. Usually we can play a game of seeing where the transgression lies in $H^*(X_1)$ by pulling this back to $H^*(K(\mathbb{Z}_2, n+1))$, and trying to gain information there.

Sometimes this will not work, section 4.2.3 and the Bockstein lemma 4.34 gives us a solution in these cases. The section is quite technical, but at least it is also helpful to us in one other way. We will also gain an understanding of the generator of $H^{k+1}(K(\mathbb{Z}_{2^i}, k))$ for $i > 1$. This will illuminate for example why our highest fibration must have fibre space $K(\mathbb{Z}_8, n+3)$.

4.2.1 Stable homotopy and preliminary results

Defining stable homotopy groups of spheres comes from the following:

Theorem 4.22 (Freudenthal suspension theorem, [1, Corollary 4.24])**.** *The suspension map $\pi_i(S^n) \to \pi_{i+1}(S^{n+1})$ is an isomorphism for $i < 2n-1$ and a surjection for $i = 2n-1$.*

If we let $i = n+k$ in the above, then we define the k-th stable homotopy group of the spheres to be the group $\pi_{n+k}(S^n)$ for n in the stable range, i.e. $n+k < 2n-1$, or $n > k+1$.

Throughout our calculations we assume n large enough so that we are in the stable range. We also assume n large enough so that the first "few" nontrivial cohomology groups of $K(\mathbb{Z}_2, n)$ consist of the relevant generators of $H^*(K(\mathbb{Z}_2, n))$, as in Serre's theorem 4.10, and not products of generators.

Theorem 4.23 (Serre's exact sequence of cohomology)**.** *Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fibration. If B and F are $(p-1)$-connected and $(q-1)$-connected respectively, then we have a long exact sequence of cohomology groups, terminating at*

$$\cdots \xrightarrow{i^*} H^{p+q-2}(F) \xrightarrow{\tau} H^{p+q-1}(B) \xrightarrow{p^*} H^{p+q-1}(E) \xrightarrow{i^*} H^{p+q-1}(F)$$

where τ is the transgression.

Sketch proof. Because of the connected-ness of B, F, from the filtration of $H^n(E)$ for $n < p+q-1$ on the E_∞ page of the associated Serre SS we get the short exact sequence $0 \to \operatorname{coker}\tau \to H^n(E) \to \ker\tau \to 0$. Using this we can get exactness in the middle line of the following:

$$\begin{tikzcd}
 & & \ker\tau \arrow[dr, hook] & \\
\cdots \arrow[r] & H^n(B) \arrow[r, "p^*"] \arrow[d, two heads] & H^n(E) \arrow[u, two heads] \arrow[r, "i^*"] & H^n(F) \arrow[r] & \cdots \\
 & \operatorname{coker}\tau \arrow[ur, hook] & &
\end{tikzcd}$$

For example we have exactness at $H^n(E)$ since $\operatorname{Im} p^* \simeq \operatorname{Im}(\operatorname{coker}\tau \hookrightarrow H^n(E)) = \ker(H^n(E) \twoheadrightarrow \ker\tau) \simeq \ker i^*$. Then we get a long exact sequence. $\square$

The long exact sequence terminates when persisting product classes begin to affect our filtration of $H^n(E)$. It follows that in the correct degrees, to find the generators of $H^n(E)$ we can just look at the kernel and cokernel of the transgression.

Now, we verify that our strategy described at the beginning of this section actually kills off the cohomology classes we want to get rid of:

Lemma 4.24. *Let G be an abelian group, X a 1-connected space, and $n \geq 2$. Consider the pullback of the path space fibration over $K(G, n+1)$ along a map $f\colon X \to K(G, n+1)$.*

$$\begin{tikzcd}
F = K(G,n) \arrow[r, "i"] & E' \arrow[r, "\pi"] \arrow[d, "p"] & * \arrow[d, "p_0"] & K(G,n) \arrow[l, "i_0"'] \\
 & X \arrow[r, "f"] & K(G,n+1) &
\end{tikzcd}$$

Then the fundamental class $\iota_F \in H^n(K(G,n); G)$ transgresses to $f^*(\iota_{n+1}) \in H^{n+1}(X; G)$ (where ι_{n+1} is the fundamental class in $H^{n+1}(K(G,n+1); G)$).

Proof. $K(G,n)$ and $K(G,n+1)$ are $(n-1)$-connected and n-connected respectively, so we have by the Serre LES of cohomology that the following is exact, where τ_0 is the transgression:

$$\cdots \to H^n(*) \xrightarrow{i_0^*} H^n(F) \xrightarrow{\tau_0} H^{n+1}(K(G,n+1)) \xrightarrow{p_0^*} H^{n+1}(*) \to \cdots$$

Therefore τ_0 is an isomorphism in this degree since the adjacent i_0^* and p_0^* are zero, and $\tau_0(\iota_F) = \iota_{n+1}$.

Since $K(G,n)$ is $n-1$-connected, and X is 1-connected, we also have the following LES of cohomology groups, where τ is the transgression of the Serre SS associated to the fibration $F = K(G,n) \to E' \to X$:

$$\cdots H^n(E') \xrightarrow{i^*} H^n(X) \xrightarrow{\tau} H^{n+1}(F) \xrightarrow{p^*} H^{n+1}(E') \to \cdots$$

Then we can stitch these together by f^* and π^*:

$$\begin{tikzcd}
\cdots \arrow[r] & H^n(*) \arrow[r, "i_0^*"] \arrow[d, "\pi^*"] & H^n(F) \arrow[r, "\tau_0"] \arrow[d, "="] & H^{n+1}(K(G,n+1)) \arrow[r, "p_0^*"] \arrow[d, "f^*"] & H^{n+1}(*) \arrow[r] \arrow[d, "\pi^*"] & \cdots \\
\cdots \arrow[r] & H^n(E') \arrow[r, "i^*"] & H^n(F) \arrow[r, "\tau"] & H^{n+1}(X) \arrow[r, "p^*"] & H^{n+1}(E') \arrow[r] & \cdots
\end{tikzcd}$$

By commutativity of the diagram, we get $\tau(\iota_F) = f^*(\tau_0(\iota_F)) = f^*(\iota_{n+1})$.

$\square$

We also need to ensure that we have maps from S^n into our approximation spaces, which we get from the following:

Proposition 4.25. *Suppose we have the following diagram,*

$$\begin{tikzcd} & E' \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & E \arrow[d, "p"] \\ Y \arrow[ur, dashed, "h"] \arrow[r, "g"] & X \arrow[r, "f"] & B \end{tikzcd}$$

where $E \xrightarrow{p} B$ is a fibration and E' is the induced fibre space of f. Then, if fg is null-homotopic, there exists a lifting of g being $h: Y \to E'$ such that $\pi_1 h = g$.

Proof. We can lift the null-homotopy of fg to $h_E: Y \to E$ such that $ph_E = fg$, due to the homotopy lifting property of the fibration $E \xrightarrow{p} B$. Then let $h = g \times h_E: Y \to X \times E$ be defined as $h(x) = (g(x), h_E(x))$. The image of this map is contained in E', since $fg(x) = ph_E(x)$. Thus $h: Y \to E'$, and clearly satisfies $\pi_1 h = g$. □

4.2.2 Serre classes

We will briefly introduce Serre classes to state theorem 4.30, which will justify our strategy of building spaces with better approximations to the $\mathbb{Z}_2$-cohomology of S^n to gain better approximations to the 2-components of the homotopy of S^n. Serre introduces the idea of a Serre class in [9].

If α is some property, then the collection of abelian groups satisfying this property is a class of abelian groups.

Definition 4.26 (Serre class)**.** Let $\mathcal{C}$ be a class of abelian groups. Suppose that for a short exact sequence of abelian groups $0 \to A \to B \to C \to 0$ we have $A, C \in \mathcal{C} \iff B \in \mathcal{C}$. Then $\mathcal{C}$ is a Serre class.

This says that the class is closed under taking subgroups, quotient groups, and group extensions. We will need the following axioms also:

1. If $A, B \in \mathcal{C}$, then $A \otimes B \in \mathcal{C}$ and $\operatorname{Tor}(A, B) \in \mathcal{C}$,
2. If $A \in \mathcal{C}$, then $A \otimes B \in \mathcal{C}$ for every abelian group B,
3. If $A \in \mathcal{C}$, then $H_n(K(A, 1); \mathbb{Z}) \in \mathcal{C}$ for every $n > 0$.

We need these to state the following generalizations of the Hurewicz theorem. We note that h a $\mathcal{C}$ isomorphism means $\ker h \in \mathcal{C}$, $\operatorname{coker} h \in \mathcal{C}$.

Theorem 4.27 (Hurewicz theorem mod $\mathcal{C}$, [9, Chapter 3, Theorem 1])**.** *Let $\mathcal{C}$ be a Serre class satisfying axioms 1 and 3. Let X be a 1-connected space. If $\pi_i(X) \in \mathcal{C}$ for all $i < n$, then $H_i(X; \mathbb{Z}) \in \mathcal{C}$ for all $i < n$ and $h: \pi_n(X) \to H_n(X; \mathbb{Z})$ is a $\mathcal{C}$-isomorphism.*

Theorem 4.28 (Relative Hurewicz theorem mod $\mathcal{C}$, [9, Chapter 3, Theorem 2])**.** *Let $\mathcal{C}$ be a Serre class satisfying axioms 2 and 3. Let X, A be 1-connected spaces, with $A \subset X$. Suppose that the inclusion map induces an epimorphism of the second homotopy groups of A, X. If $\pi_i(X, A) \in \mathcal{C}$ for all $i < n$, then $H_i(X, A; \mathbb{Z}) \in \mathcal{C}$ for all $i < n$ and $h: \pi_n(X, A) \to H_n(X, A; \mathbb{Z})$ is a $\mathcal{C}$-isomorphism.*

Example 4.29. The class $\mathcal{C}_2$ of torsion abelian groups with the order of every element odd is a Serre class, that also satisfies axioms $1, 2, 3$, hence we have the Hurewicz theorems mod $\mathcal{C}_2$.

Theorem 4.30 ([8, Chapter 10, Theorem 4])**.** *Let $f: X \to Y$ be a map of simply-connected spaces with the cohomology of X finitely generated, such that $f^*: H^i(Y; \mathbb{Z}_2) \to H^i(X; \mathbb{Z}_2)$ is an isomorphism for $i < n$ and a monomorphism for $i = n$. Then $\pi_i(X)$ and $\pi_i(Y)$ have isomorphic 2-components for $i < n$.*

Sketch proof. WLOG assume f is an inclusion. By duality we get f_* isomorphic for $i < n$ and epimorphic in degree n. Then $H_i(X, A; \mathbb{Z}_2) = 0$ for $i \leq n$, and we claim that this implies $H_i(X, A; \mathbb{Z}) \in \mathcal{C}_2$ for $i \leq n$. Then by the relative Hurewicz theorem we have $\pi_i(X, A) \in \mathcal{C}_2$ for $i \leq n$. By the exact homotopy sequence mod $\mathcal{C}_2$, we get that $f_\#: \pi_i(A) \to \pi_i(X)$ is $\mathcal{C}_2$-isomorphic for $i < n$, and epimorphic in degree n. We claim this implies $\pi_i(A)$ and $\pi_i(X)$ have isomorphic 2-components for $i < n$. □

4.2.3 Bockstein homomorphisms

Next, we gain a better understanding of the cohomology class that generates $H^{n+1}(K(\mathbb{Z}_{2^i}, n); \mathbb{Z}_2) = \mathbb{Z}_2$, which for $n = 2$ we saw in example 4.13. It turns out that these cohomology classes are related to the Bockstein differentials in the Bockstein spectral sequence (for prime $p = 2$). Then we will use this to prove a result known as the Bockstein lemma, which will aid us in our calculations for the homotopy groups of the spheres.

We will not describe the Bockstein spectral sequence here, and refer to [5, Chapter 7]. Recall that the Bockstein spectral sequence comes from considering the following exact couple (meaning the diagram is exact in each spot):

$$\begin{tikzcd}
H^*(-;\mathbb{Z}) \arrow[rr, "\cdot 2"] & & H^*(-;\mathbb{Z}) \arrow[dl, "\rho"] \\
& H^*(-;\mathbb{Z}_2) \arrow[ul, "\beta"] &
\end{tikzcd}$$

with β the connecting homomorphism of the long exact sequence of cohomology induced by $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_2$ (see example 1.18), and ρ is the reduction of coefficients mod 2. Then the first differential is $d_1 = \rho\beta = \mathrm{Sq}^1$.

Higher differentials are defined on groups E^r such that E^r is a sub-quotient of E^{r-1}, and $E^1 = H^*(-;\mathbb{Z}_2)$. We can also consider d_i defined on $\bigcap_{r=1}^{i-1} \ker d_r \subset H^*(-;\mathbb{Z}_2)$ and allow for some indeterminancy. If x is the reduction mod 2 of some class with coefficients in 2^{i+1}, then $d_i(x) = 0$. We therefore have that the map $d_i\rho\colon H^*(-;\mathbb{Z}_{2^i}) \to H^{*+1}(-;\mathbb{Z}_2)$ is well-defined. This is also natural, since the Bockstein differentials are, and so we have by corollary 1.23 there is a corrsesponding map which we'll denote by $d_i\rho\colon K(\mathbb{Z}_{2^i}, n) \to K(\mathbb{Z}_2, n+1)$.

We can use this map to construct a fibration corresponding to that induced by the exact sequence $0 \to \mathbb{Z}_2 \to \mathbb{Z}_{2^{i+1}} \to \mathbb{Z}_{2^i} \to 0$:

Lemma 4.31. *The map $d_i\rho\colon K(\mathbb{Z}_{2^i}, n) \to K(\mathbb{Z}_2, n+1)$ and path space fibration over $K(\mathbb{Z}_2, n+1)$ induces a pullback fibration where $E' = K(\mathbb{Z}_{2^{i+1}}, n)$, and the fibration $K(\mathbb{Z}_2, n) \to K(\mathbb{Z}_{2^{i+1}}, n) \to K(\mathbb{Z}_{2^i}, n)$ corresponds to that induced by the exact sequence $0 \to \mathbb{Z}_2 \xrightarrow{\cdot 2^i} \mathbb{Z}_{2^{i+1}} \to \mathbb{Z}_{2^i} \to 0$.*

$$\begin{tikzcd}
K(\mathbb{Z}_2, n) \arrow[r, "j"] & E' \arrow[r] \arrow[d, "p"] & * \arrow[d] \\
& K(\mathbb{Z}_{2^i}, n) \arrow[r, "d_i\rho"] & K(\mathbb{Z}_2, n+1)
\end{tikzcd}$$

Proof. From the long exact sequence of homotopy groups, $E' = K(G, n)$ is clear and we have $0 \to \mathbb{Z}_2 \to G \to \mathbb{Z}_{2^i} \to 0$. We need to verify that $G = \mathbb{Z}_{2^{i+1}}$ rather than $G = \mathbb{Z}_2 \oplus \mathbb{Z}_{2^i}$. But from the Serre spectral sequence we have that $\iota_n \in H^n(K(\mathbb{Z}_2, n))$ transgresses to the cohomology class in $H^{n+1}(K(\mathbb{Z}_{2^i}, n))$ reprsenting $d_i\rho$ by lemma 4.24. Therefore $H^n(E'; \mathbb{Z}_2) = \mathbb{Z}_2$ (only one term in the filtration) and using the universal coefficient theorem this shows that G must be cyclic, so $G = \mathbb{Z}_{2^{i+1}}$. Then the non null-homotopic maps of the induced fibration represent non-trivial homomorphisms between the coefficient groups as in example 4.4, and so correspond to the only non-trivial ones. □

Remark 4.32. In this result we described the cohomology class that generates $H^{n+1}(K(\mathbb{Z}_{2^i}, n))$. When dealing with $H^*(K(\mathbb{Z}_{2^i}, n))$, we suppress the reduction mod 2 of the fundamental class and write ι_n for the generator of $H^n(K(\mathbb{Z}_{2^i}, n))$. Correspondingly we will write $d_i\iota_n$ for the generator of $H^{n+1}(K(\mathbb{Z}_{2^i}, n))$ that represents the map $d_i\rho$.

We now describe an explicit result relating to this fibration, to prove a more general result known as the Bockstein lemma.

Proposition 4.33. *Let $F \xrightarrow{j} E \xrightarrow{p} B$ be the fibration $K(\mathbb{Z}_2, n) \xrightarrow{j} K(\mathbb{Z}_{2^{i+1}}, n) \xrightarrow{p} K(\mathbb{Z}_{2^i}, n)$ constructed in the previous lemma. Consider the associated Serre spectral sequence, and let τ*

denote the transgression. Then for $u = \iota_F \in H^n(F;\mathbb{Z}_2)$ the fundamental class, and $v = \rho(\iota_B)$ the reduction mod 2 of the fundamental class of B, we have $\tau(u) = d_i v$, and moreover $j^* d_{i+1} p^*(v) = \operatorname{Sq}^1(u)$.

Proof. We consider Serre's LES of cohomology from theorem 4.23

$$\begin{array}{ccccccccccccc} 0 & \longrightarrow & H^n(B) & \xrightarrow{p^*} & H^n(E) & \xrightarrow{j^*} & H^n(F) & \xrightarrow{\tau} & H^{n+1}(B) & \xrightarrow{p^*} & H^{n+1}(E) & \xrightarrow{j^*} & H^{n+1}(F) \\ & & v & & \iota_E & & u & & d_i(v) & & d_{i+1}(\iota_E) & & d_1(u) \end{array}$$

each group is isomorphic to $\mathbb{Z}_2$ and we write generators beneath as described thereom 4.10.

We note that d_i is zero on $H^n(E)$ and so $d_{i+1}p^*(v)$ is defined. By exactness we get the first p^* is monomorphic, and so isomorphic, and then the first j^* is 0. The first transgression τ is isomorphic also, and $\tau(u) = (d_i\rho)^*(\iota) = d_i\rho(\iota_\beta) = d_i(v)$. By exactness again the second p^* is zero and so the second j^* is injective and hence isomorphic, so that we have $j^* d_{i+1} p^*(v) = j^* d_{i+1}(\iota_E) = \operatorname{Sq}^1(u)$. □

This lemma shows a relation between the transgression of an element and applying Sq^1 to it. Now we consider the general case, and prove it by mapping into the specific example desribed in the above proposition.

Theorem 4.34 (Bockstein lemma, [8, Chapter 11, Theorem 1])**.** *Let $F \xrightarrow{j} E \xrightarrow{p} B$ be a fibration. Suppose that for some $u \in H^n(F;\mathbb{Z}_2)$ that is transgressive, we have $d_i(v) = \tau(u)$ for $v \in H^n(E;\mathbb{Z}_2)$. Then we have*

$$j^* d_{i+1} p^*(v) = \operatorname{Sq}^1(u)$$

Proof. Since $d_i(v) = \tau(u)$, we have by exactness and naturality that $0 = p^*(\tau(u)) = p^* d_i(v) = d_i p^*(v)$, so that $d_{i+1}p^*(v)$ is well-defined. As $d_i(v)$ is defined, we have $v \in \bigcap_{r=1}^{i-1} \ker d_r$ and in particular $d_{i-1}(v) = 0$, so that we can consider v as the reduction mod 2 of a class $w \in H^n(B;\mathbb{Z}_{2^i})$ as indicated by the map in the diagram. Moreover the triangle involving w, v, ρ commutes up to homotopy, as does the larger triangle because both directions act the same on the fundamental class of $K(\mathbb{Z}_2, n+1)$ (so they represent the same cohomology classes).

$$\begin{tikzcd}
F \arrow[r, "j"] \arrow[dd, dashed, "h"] & E \arrow[r, "p"] \arrow[dd, dashed, "g"] & B \arrow[dr, "v"] \arrow[drr, "d_i v"] \arrow[dd, dashed, "w"] & & \\
& & & K(\mathbb{Z}_2, n) & K(\mathbb{Z}_2, n+1) \\
F_0 = K(\mathbb{Z}_2, n) \arrow[r, "j_0"] & E_0 = K(\mathbb{Z}_{2^{i+1}}, n) \arrow[r, "p_0"] & B_0 = K(\mathbb{Z}_{2^i}, n) \arrow[ur, "\rho"] \arrow[urr, "d_i\rho"'] & &
\end{tikzcd}$$

Since the composition $(d_i v)p$ is null-homotopic, we have by commutativity of the diagram that the composition $(d_i\rho)wp$ is null-homotopic also. It follows by lemma 4.25 that we can lift wp to a map $g\colon E \to E_0$ such that $p_0 g = wp$. Since pj is a constant map, we have p_0 constant on $\operatorname{Im} gj$, and so $\operatorname{Im} gj$ is contained in F_0, so we can consider this as a map $h\colon F \to F_0$ so that the left square of the diagram commutes also.

By our construction, $\tau(h) = w^*(\tau_0(\iota_{F_0}))$, where τ_0 refers to the transgression in the Serre SS associated to $F_0 \to E_0 \to B_0$. From proposition 4.33 we have $\tau_0(\iota_{F_0}) = d_i\rho$ and so $\tau(h) = w^* d_i \rho = d_i \rho w = d_i v = \tau(u)$. We do not know whether h is u, but they are at least equal modulo the kernel of τ, and $\operatorname{Sq}^1 h \equiv \operatorname{Sq}^1 u$ modulo $\operatorname{Sq}^1(\ker\tau) = \operatorname{Sq}^1(\operatorname{Im} j^*)$ since the squares commute with transgression.

The upshot of considering h is that we can compute $j^* d_{i+1} p^*(v)$ by applying h^* to the result $j_0^* d_{i+1}(\rho(\iota_{B_0})) = \operatorname{Sq}^1(\iota_{F_0})$ of proposition 4.33. We have

$$\begin{aligned} h^* \operatorname{Sq}^1(\iota_{F_0}) &= h^* j_0^* d_{i+1} p_0^* v_0 \\ &= (j_0 h)^* d_{i+1}(v_0 p_0) \\ &= (gj)^* d_{i+1}(v_0 p_0) \\ &= j^* d_{i+1} v_0 p_0 g_0 \\ &= j^* d_{i+1} v p \\ &= j^* d_{i+1} p^* v \end{aligned}$$

and $h^* \operatorname{Sq}^1(\iota_{F_0}) = \operatorname{Sq}^1 h$, completing the result. □

4.2.4 Computations

Assume n large enough so that we are in the stable range. We aim to construct spaces that approximate the homotopy of S^n. The following calculations are carried out in [8, Chapter 12].

Step 1: $\pi_n(S^n) = \mathbb{Z}$

Let $f\colon S^n \to K(\mathbb{Z}, n)$ represent the homotopy class of a generator of $\pi_n(K(\mathbb{Z}, n)) = \mathbb{Z}$. Then f^* is isomorphic through degree n and monomorphic in degree $n+1$ (which is easy to see since $H^{n+1}(K(\mathbb{Z}, n)) = 0$). Therefore by theorem 4.30, $f_\#\colon \pi_i(S^n) \to \pi_i(K(\mathbb{Z}, n))$ is isomorphic through degree n, so that $\pi_n(S^n) = \mathbb{Z}$, as expected.

Step 2: $\pi_{n+1}(S^n) = \mathbb{Z}_2$

Recall from theorem 4.10 that $H^{n+1}(K(\mathbb{Z}, n)) = 0$ and $H^{n+2}(K(\mathbb{Z}, n)) = \mathbb{Z}_2$, generated by the class $\operatorname{Sq}^2 \iota_n$ where ι_n is reduction mod 2 of the fundamental class of $K(\mathbb{Z}, n)$. Let $\operatorname{Sq}^2\colon K(\mathbb{Z}, n) \to K(\mathbb{Z}_2, n+2)$ be the map (up to homotopy) representing $\operatorname{Sq}^2 \iota_n$.

We want to build a better approximation X_1 and a map $f_1\colon S^n \to X_1$ such that f_1^* induces an isomorphism through degree $n+1$ and a monomorphism on degree $n+2$ (we will aim for $H^{n+2}(X_1; \mathbb{Z}_2) = 0$ to ensure this).

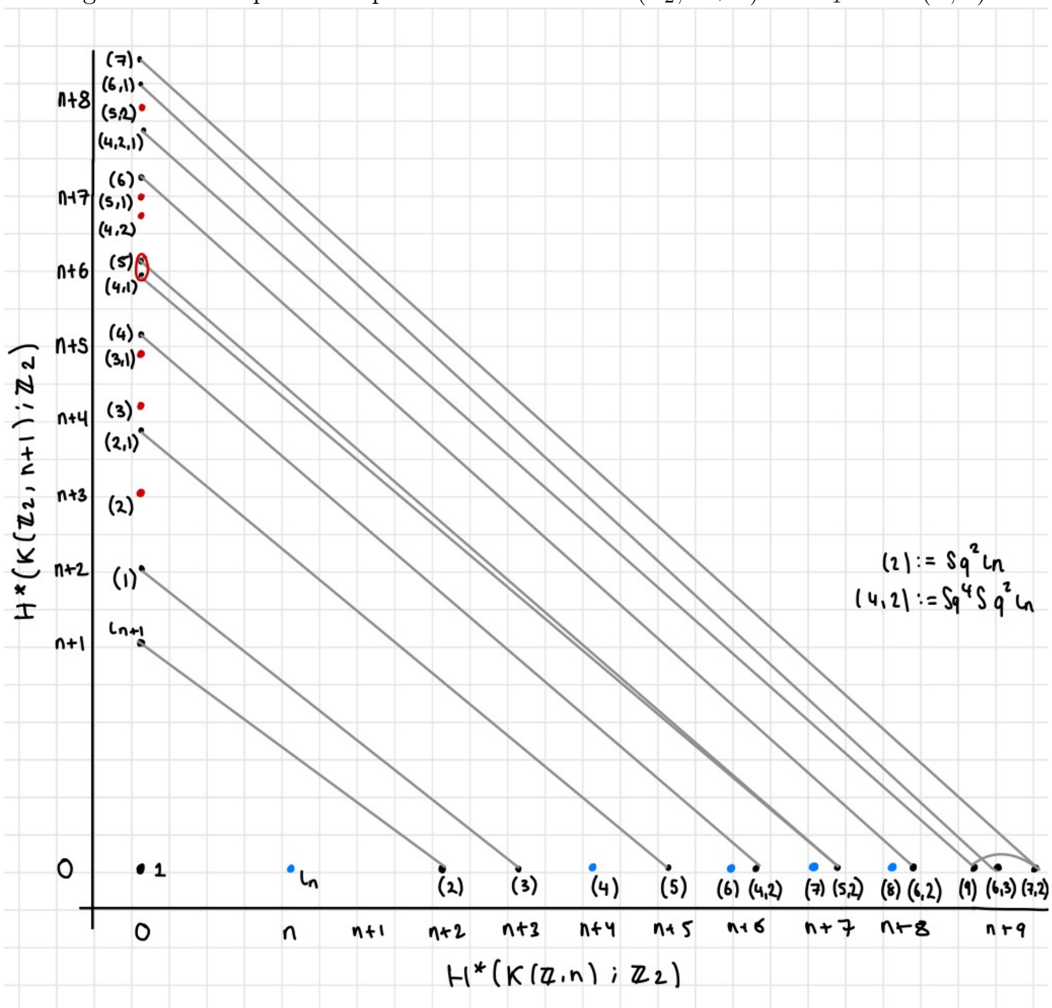
The composition $\operatorname{Sq}^2 f\colon S^n \to K(\mathbb{Z}_2, n+2)$ is null-homotopic since $\pi_n(K(\mathbb{Z}_2, n+2)) = 0$. Therefore if we consider the path space fibration $K(\mathbb{Z}_2, n+1) \to * \to K(\mathbb{Z}_2, n+2)$, we yield the following by Proposition 4.25:

$$\begin{tikzcd} & K(\mathbb{Z}_2, n+1) \arrow[r, "i_1"] & X_1 \arrow[r] \arrow[d, "p_1"] & * \arrow[d] \\ S^n \arrow[urr, "f_1"] \arrow[rr, "f"] & & K(\mathbb{Z}, n) \arrow[r, "\operatorname{Sq}^2"] & K(\mathbb{Z}_2, n+2) \end{tikzcd}$$

We need to verify that $H^{n+2}(X_1) = 0$. If this is the case, then $\pi_{n+1}(X_1) = \pi_{n+1}(S^n)$, and from the long exact sequence of homotopy groups for $K(\mathbb{Z}_2, n+1) \to X_1 \to K(\mathbb{Z}, n)$, we obtain $\pi_{n+1}(X_1) = \mathbb{Z}_2$.

We compute some of the cohomology groups of X_1 through use of the Serre spectral sequence. We include the calculation of higher cohomology groups of X_1, to use for further calculations. The E_2 page is the tensor product of $H^*(K(\mathbb{Z}, n))$ and $H^*(K(\mathbb{Z}_2, n+2))$ which we know from theorem 4.10.

Figure 1: Serre spectral sequence associated to $K(\mathbb{Z}_2, n+1) \xrightarrow{i_1} X_1 \xrightarrow{p_1} K(\mathbb{Z}, n)$.



The admissible sequences have already been written out in example 3.7, which we can use to write $H^*(K(\mathbb{Z}, n))$ and $H^*(K(\mathbb{Z}_2, n+2))$. The fundamental class ι_{n+1} transgresses to $\mathrm{Sq}^2 \iota_n$ by lemma 4.24. Since the squares commute with the transgression, the rule for computing the transgression is $\tau((\cdot)\iota_{n+1}) = (\cdot)\,\mathrm{Sq}^2 \iota_n$. Here, these computations are just exercises in applying Adém relations. We let the red dots on our diagram denote the classes in the kernel of the transgression, and the blue dots denote the classes in the cokernel of the transgression.

By Serre's long exact sequence 4.23, to obtain the cohomology of X_1 we just need to look at the kernel and cokernel of the transgression. A few remarks on the calculation are as follows:

- $\mathrm{Sq}^2 \iota_{n+1} \in \ker \tau$ since $\tau(\mathrm{Sq}^2 \iota_{n+1}) = \mathrm{Sq}^2 \mathrm{Sq}^2 \iota_n = \mathrm{Sq}^1 \mathrm{Sq}^2 \mathrm{Sq}^1 \iota_n = 0$ (see example 3.8) as $\mathrm{Sq}^1 \iota_n = 0$ for $\iota_n \in H^*(K(\mathbb{Z}, n))$. Therefore we get a generator α of $H^{n+3}(X_1)$ such that $i_1^*(\alpha) = \mathrm{Sq}^2 \iota_{n+1}$.
- Correspondingly, $\mathrm{Sq}^4 \iota_n \in \operatorname{coker} \tau$ since $\mathrm{Sq}^2 \iota_{n+1}$ is the only generator of $H^{n+3}(K(\mathbb{Z}_2, n+1))$. We get a generator of $H^{n+4}(X_1)$, namely $p_1^* \mathrm{Sq}^4 \iota_n$.
- The other classes in $\ker \tau$ follow from applying Adém relations. In particular, $\tau(\mathrm{Sq}^5 \iota_{n+1}) = \mathrm{Sq}^5 \mathrm{Sq}^2 \iota_n = \tau(\mathrm{Sq}^4 \mathrm{Sq}^1 \iota_{n+1})$, so that $\mathrm{Sq}^5 \iota_{n+1} + \mathrm{Sq}^4 \mathrm{Sq}^1 \iota_{n+1} \in \ker \tau$, and we write the generator $\delta = \delta(5+4, 1) \in H^{n+6}(X_1)$ such that $i_1^*(\delta(5+4,1)) = \mathrm{Sq}^5 \iota_{n+1} + \mathrm{Sq}^4 \mathrm{Sq}^1 \iota_{n+1}$.
- In degree $n+8$ we have $\mathrm{Sq}^4 \mathrm{Sq}^2 \mathrm{Sq}^1 \mathrm{Sq}^2 \iota_n = \mathrm{Sq}^7 \mathrm{Sq}^2 \iota_n + \mathrm{Sq}^9 \iota_n$.

We find the cohomology of X_1 up to degree $n+8$ is as follows, where we write generators, with for example $\epsilon = \epsilon(5,1)$ representing the cohomology class such that $i_1^*(\epsilon) = \mathrm{Sq}^5\,\mathrm{Sq}^1\,\iota_{n+1}$. This will help us keep track of where our cohomology classes "came from" in later calculations.

k	$H^{n+k}(X_1)$	k	$H^{n+k}(X_1)$
0	$p_1^*\iota_n$	5	$\gamma(3,1)$
1	0	6	$p_1^*\,\mathrm{Sq}^6\,\iota_n,\ \delta(5+4,1)$
2	0	7	$p_1^*\,\mathrm{Sq}^7\,\iota_n,\ \epsilon(5,1),\ \zeta(4,2)$
3	$\alpha(2)$	8	$p_1^*\,\mathrm{Sq}^8\,\iota_n,\ \eta(5,2)$
4	$p_1^*\,\mathrm{Sq}^4\,\iota_n,\ \beta(3)$		

In particular, we have verified that $H^{n+2}(X_1)$ is zero.

Step 3: $\pi_{n+2}(S^n) = \mathbb{Z}_2$

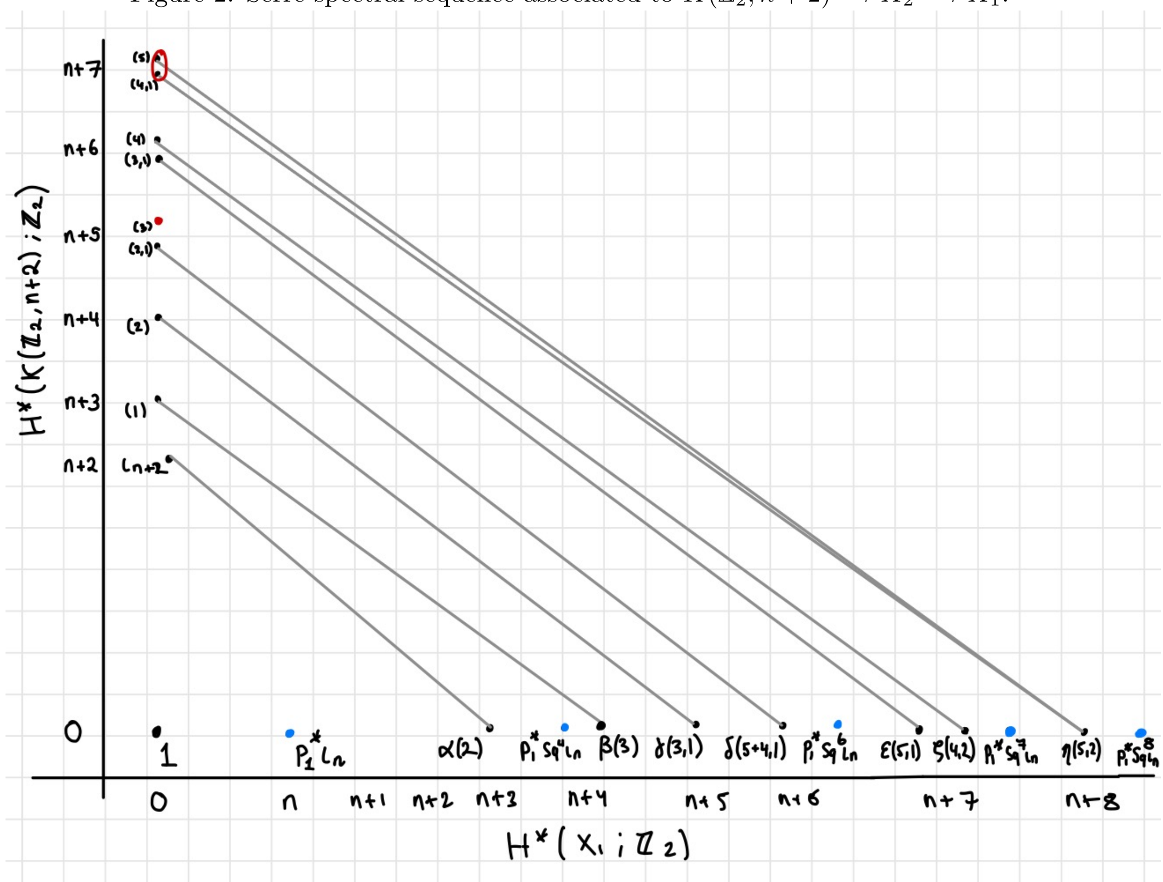
Once more we represent $\alpha \in H^{n+3}(X_1)$ as a map $\alpha\colon X_1 \to K(\mathbb{Z}_2, n+3)$. We repeat our induced fibration construction, to obtain a new approximating space X_2, where we can lift $f_1\colon S^n \to X_1$ to $f_2\colon S^n \to X_2$ by proposition 4.25 since the composition αf_1 is null-homotopic.

$$\begin{tikzcd}
 & K(\mathbb{Z}_2, n+2) \arrow[r, "i_2"] & X_2 \arrow[r] \arrow[d, "p_2"] & * \arrow[d] \\
 & & X_1 \arrow[r, "\alpha"] \arrow[d] & K(\mathbb{Z}_2, n+3) \\
S^n \arrow[rruu, "f_2"] \arrow[rr, "f"] & & K(\mathbb{Z}, n) &
\end{tikzcd}$$

We expect $H^{n+3}(X_2)$ to be zero, so that the map of homotopy groups induced by f_2 is isomorphic in degree $n+2$ and $\pi_{n+2}(S^n) = \pi_{n+2}(X_2) = \mathbb{Z}_2$.

We use the Serre spectral sequence to compute cohomology groups of $H^*(X_2)$. Since the base X_1 does not have its cohomology written in terms of Steenrod squares, the calculations of the transgression are more awkward, but we describe the first few:

Figure 2: Serre spectral sequence associated to $K(\mathbb{Z}_2, n+2) \xrightarrow{i_2} X_2 \xrightarrow{p_2} X_1$.



- $\tau(\iota_{n+2}) = \alpha$ by proposition 4.24.
- $\tau(\operatorname{Sq}^1 \iota_{n+2}) = \operatorname{Sq}^1 \alpha$. We have $i_1^*(\operatorname{Sq}^1 \alpha) = \operatorname{Sq}^1 \iota_1^*(\alpha) = \operatorname{Sq}^1 \operatorname{Sq}^2 \iota_{n+1} = \operatorname{Sq}^3 \iota_{n+1} = i_1^*(\beta)$. However $i_1^*(p_1^* \operatorname{Sq}^4 \iota_n) = 0$ by exactness. Therefore $\operatorname{Sq}^1 \alpha = \beta + ()p_1^* \operatorname{Sq}^4 \iota_n$ for $()$ some undetermined coefficient. In any case, the cokernel of τ in this degree is generated by $p_1^* \operatorname{Sq}^4 \iota_n$, so we get a generator of $H^{n+4}(X_2)$, namely $p_2^* p_1^* \operatorname{Sq}^4 \iota_n$.
- $i_1^*(\tau(\operatorname{Sq}^2 \iota_{n+2})) = \operatorname{Sq}^2 i_1^*(\alpha) = \operatorname{Sq}^2 \operatorname{Sq}^2 \iota_{n+1} = \operatorname{Sq}^3 \operatorname{Sq}^1 \iota_{n+1} = i_1^*(\gamma)$. Since γ is the only generator of $H^{n+5}(X_1)$, we have $\tau(\operatorname{Sq}^2 \iota_{n+2}) = \gamma$.
- $\tau(\operatorname{Sq}^3 \iota_{n+2}) = \operatorname{Sq}^1 \tau(\operatorname{Sq}^2 \iota_{n+2}) = \operatorname{Sq}^1 \gamma$. We show in lemma 4.35 that $\operatorname{Sq}^1 \gamma = 0$, so that $\operatorname{Sq}^3 \iota_{n+2} \in \ker \tau$ and we obtain $A \in H^{n+5}(X_2)$ such that $i_2^*(A) = \operatorname{Sq}^3 \iota_{n+2}$.
- $i_1^*(\tau(\operatorname{Sq}^2 \operatorname{Sq}^1 \iota_{n+2})) = \operatorname{Sq}^2 \operatorname{Sq}^1 \operatorname{Sq}^2 \iota_{n+1} = \operatorname{Sq}^4 \operatorname{Sq}^1 \iota_{n+1} + \operatorname{Sq}^5 \iota_{n+1} = i_1^*(\delta)$. We have $i_1^*(p_1^* \operatorname{Sq}^6 \iota_n) = 0$, so $\tau(\operatorname{Sq}^2 \operatorname{Sq}^1 \iota_{n+2}) = \delta + ()p_1^* \operatorname{Sq}^6 \iota_n$, but since $\tau(\operatorname{Sq}^3 \iota_{n+2}) = 0$, we can conclude that the $p_1^* \operatorname{Sq}^6 \iota_n \in \operatorname{coker} \tau$ and so we obtain a generator $p_2^* p_1^* \operatorname{Sq}^6 \iota_n$ of $H^{n+6}(X_2)$.
- $i_1^*(\tau(\operatorname{Sq}^4 \iota_n)) = i_1^*(\operatorname{Sq}^4 \alpha) = \operatorname{Sq}^4 \operatorname{Sq}^2 \iota_{n+1} = i^*(\zeta)$ and $i_1^*(\tau(\operatorname{Sq}^3 \operatorname{Sq}^1 \iota_n)) = \operatorname{Sq}^3 \operatorname{Sq}^1 \operatorname{Sq}^2 \iota_{n+1} = \operatorname{Sq}^5 \operatorname{Sq}^1 \iota_{n+1} = i_1^*(\epsilon)$. It follows that $\tau(\operatorname{Sq}^4 \iota_n) = \zeta + ()p_1^* \operatorname{Sq}^7 \iota_n$ and $\tau(\operatorname{Sq}^3 \operatorname{Sq}^1 \iota_n) = \epsilon + ()p_1^* \operatorname{Sq}^7 \iota_n$.

  Regardless of the undertermined coefficients, we have that $\operatorname{coker} \tau$ is generated by $p_1^* \operatorname{Sq}^7 \iota_n$, and we obtain a generator $p_2^* p_1^* \operatorname{Sq}^7 \iota_n$ of $H^{n+7}(X_2)$.
- We have $\tau(\operatorname{Sq}^4 \operatorname{Sq}^1 \iota_{n+2}) = \operatorname{Sq}^4 \beta$, and $i_1^*(\operatorname{Sq}^4 \beta) = \operatorname{Sq}^4 \operatorname{Sq}^3 \iota_n = \operatorname{Sq}^5 \operatorname{Sq}^2 \iota_n = i_1^*(\eta)$. Also, $\tau(\operatorname{Sq}^5 \iota_{n+2}) = \tau(\operatorname{Sq}^1 \operatorname{Sq}^4 \iota_{n+2}) = \operatorname{Sq}^1 \zeta$, with $i_1^*(\operatorname{Sq}^1 \zeta) = \operatorname{Sq}^1 \operatorname{Sq}^4 \operatorname{Sq}^2 \iota_n = \operatorname{Sq}^5 \operatorname{Sq}^1 \iota_n = i_1^*(\eta)$. Thus $\tau(\operatorname{Sq}^4 \operatorname{Sq}^1 \iota_{n+2}) = \eta + ()p_1^* \operatorname{Sq}^8 \iota_n$ and $\tau(\operatorname{Sq}^5 \iota_{n+2}) = \eta + ()p_1^* \operatorname{Sq}^8 \iota_n$.

We need to determine whether their sum is in $\ker\tau$ or not. But from the Adém relation $\operatorname{Sq}^2\operatorname{Sq}^3 = \operatorname{Sq}^4\operatorname{Sq}^1 + \operatorname{Sq}^5$, we have $\tau(\operatorname{Sq}4\operatorname{Sq}^1\iota_{n+2} + \operatorname{Sq}^5\iota_{n+2}) = \tau(\operatorname{Sq}^2\operatorname{Sq}^3\iota_{n+2}) = 0$ since $\tau(\operatorname{Sq}^3\iota_{n+2}) = 0$. Therefore we write $B(5+4,1) \in H^{n+7}(X_1)$ such that $\iota_2^*(B) = \operatorname{Sq}^4\operatorname{Sq}^1\iota_{n+2} + \operatorname{Sq}^5\iota_{n+2}$.

Thus the cohomology of X_2 up to degree $n+7$ is as follows:

k	0	1	2	3	4	5	6	7
$H^{n+k}(X_2)$	$p_2^*p_1^*\iota_n$	0	0	0	$p_2^*p_1^*\operatorname{Sq}^4\iota_n$	$A(3)$	$p_2^*p_1^*\operatorname{Sq}^6\iota_n$	$p_2^*p_1^*\operatorname{Sq}^7\iota_n$, $B(5+4,1)$

We have verified that $H^{n+3}(X_2) = 0$.

Step 4: $\pi_{n+3}(S^n) = \mathbb{Z}_8$, $\pi_{n+4}(S^n) = \pi_{n+5}(S^n) = 0$

We want to kill off $\operatorname{Sq}^4\iota_n \in H^{n+4}(X_2)$. If we repeat our construction, obtaining a fibration $K(\mathbb{Z}_2, n+3) \to X_3 \to X_2$ such that ι_{n+3} transgresses to $p_2^*p_1^*\operatorname{Sq}^4\iota_n \in H^{n+4}(X_2)$, then we would have $\tau(\operatorname{Sq}^1\iota_{n+3}) = \operatorname{Sq}^1 p_2^*p_1^*\operatorname{Sq}^4\iota_n = p_2^*p_1^*\operatorname{Sq}^5\iota_n = p_2^*p_1^*\tau(\operatorname{Sq}^2\operatorname{Sq}^1\iota_{n+1}) = 0$ since $p_1^*\tau = 0$. Therefore $\operatorname{Sq}^1\iota_{n+3} \in \ker\tau$, so we have non-trivial cohomology in degree $n+4$. Thus we would gain "nothing new" from this new space X_3.

Moreover, in lemma 4.35 we will show that $d_3(p_2^*p_1^*\operatorname{Sq}^4\iota_n) = A$. Therefore constructing a fibration $K(\mathbb{Z}_4, n+3) \to X_3 \to X_2$ such that ι_{n+3} transgresses to $p_2^*p_1^*\operatorname{Sq}^4\iota_n$ would not work either, since $d_2\iota_{n+3} \in H^{n+1}(K(\mathbb{Z}_4, n+3))$ would then transgress to $d_2(p_2^*p_1^*\operatorname{Sq}^4\iota_n) = 0$.

Clearly, the correct thing to do is construct the fibration $K(\mathbb{Z}_8, n+3) \to X_3 \to X_2$ with $\tau(\iota_{n+3}) = p_2^*p_1^*\operatorname{Sq}^4\iota_n$, so that $\tau(d_3\iota_{n+3}) = A$. Since $d_2(p_2^*p_1^*\operatorname{Sq}^4\iota_n) = 0$, it follows that $p_2^*p_1^*\operatorname{Sq}^4\iota_n \in H^{n+4}(X_2)$ represents the mod 2 reduction of a class which we'll continue to call $p_2^*p_1^*\operatorname{Sq}^4\iota_n \in H^{n+4}(X_2;\mathbb{Z}_8)$, and we use the map $p_2^*p_1^*\operatorname{Sq}^4\iota_n\colon X_2 \to K(\mathbb{Z}_8, n+4)$ to construct an induced fibration as follows:

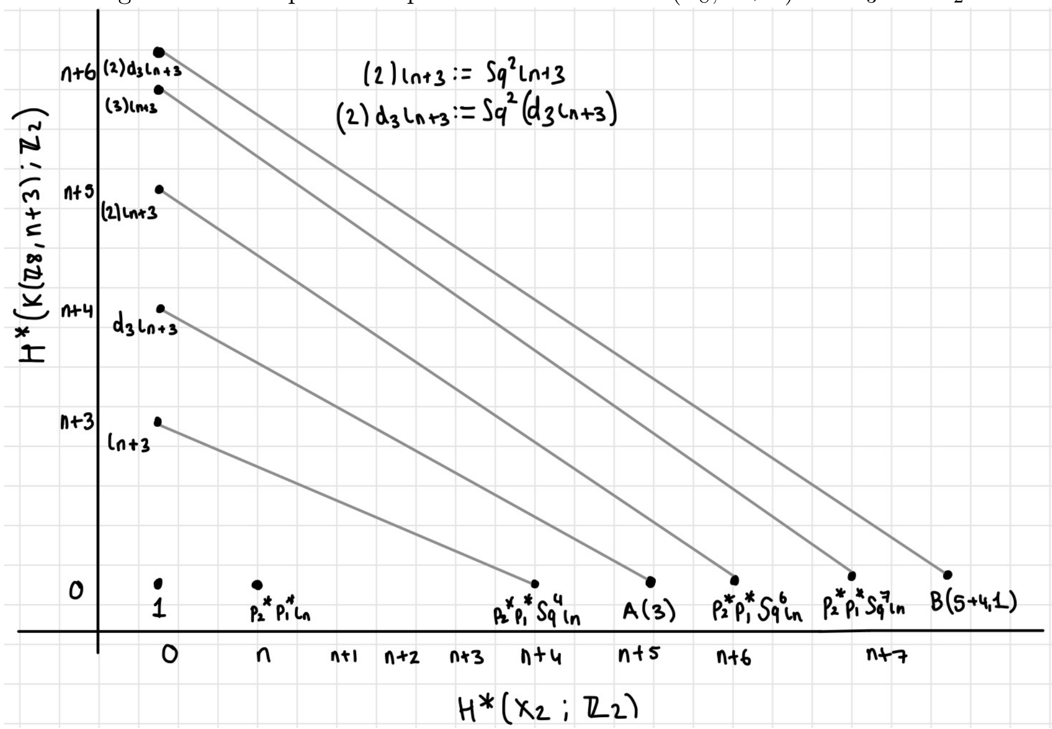
$$\begin{tikzcd}
 & K(\mathbb{Z}_8, n+3) \arrow[r, "i_3"] & X_3 \arrow[r] \arrow[d, "p_3"] & * \arrow[d] \\
 & & X_2 \arrow[r, "p_2^*p_1^*\operatorname{Sq}^4\iota_n"] \arrow[d] & K(\mathbb{Z}_8, n+4) \\
 & & X_1 \arrow[d] & \\
S^n \arrow[rruuu, "f_3"] \arrow[rr, "f"] & & K(\mathbb{Z}, n) &
\end{tikzcd}$$

If we verify that $H^{n+4}(X_3) = 0$, then f_3 induces an isomorphsim of cohomology groups up to degree $n+3$, and a monomorphism in degree $n+4$. Then $\pi_{n+3}(S^n) = \pi_{n+3}(X_3) = \mathbb{Z}_8$ from the long exact sequence of homotopy groups.

The $\mathbb{Z}_2$-cohomology of $K(\mathbb{Z}_8, n+3)$ is described in theorem 4.10. We verify that $H^{n+4}(X_3) = 0$, and in fact that $H^{n+5}(X_3) = H^{n+6}(X_3) = 0$. The first two transgressions have been detailed. We also have

- $\tau(\operatorname{Sq}^2\iota_{n+3}) = \operatorname{Sq}^2(p_2^*p_1^*\operatorname{Sq}^4\iota_n) = p_2^*p_1^*\operatorname{Sq}^6\iota_n$,
- $\tau(\operatorname{Sq}^3\iota_{n+3}) = \operatorname{Sq}^3(p_2^*p_1^*\operatorname{Sq}^4\iota_n) = p_2^*p_1^*\operatorname{Sq}^7\iota_n$,
- $\tau(\operatorname{Sq}^2(d_3\iota_{n+3})) = \operatorname{Sq}^2(A)$, and $i_2^*(\operatorname{Sq}^2 A) = \operatorname{Sq}^2\operatorname{Sq}^3\iota_{n+3} = \operatorname{Sq}^4\operatorname{Sq}^1\iota_{n+3} + \operatorname{Sq}^5\iota_{n+3} = i_2^*(B)$. Thus we get $\operatorname{Sq}^2(d_3\iota_{n+3}) = B + ()p_2^*p_1^*\operatorname{Sq}^7\iota_n$. In any case, there is trivial kernel and cokernel for the transgression in this degree.

Figure 3: Serre spectral sequence associated to $K(\mathbb{Z}_8, n+3) \xrightarrow{i_3} X_3 \xrightarrow{p_3} X_2$.



Since $H^{n+5}(X_3) = H^{n+6}(X_3) = 0$, f_3 induces an isomorphism of cohomology groups up to degree $n+5$, and a monomorphism in degree $n+6$. Therefore from the long exact sequence of homotopy groups we get $\pi_{n+3}(S^n) = \mathbb{Z}_8$, and $\pi_{n+4}(S^n) = \pi_{n+5}(S^n) = 0$.

Though it is not included in the diagrams, we find that $H^{n+8}(X_3) = \mathbb{Z}_2$, generated by $p_3^* p_2^* p_1^* \operatorname{Sq}^8 \iota_n$. This is the next cohomology class that needs to be killed, but we will stop here.

Lastly, we prove the claims we made during our calculations, namely,

Lemma 4.35.

1. *In* $H^*(X_1)$, $\operatorname{Sq}^1 \gamma = 0$,
2. *In* $H^*(X_2)$, $d_3(p_2^* p_1^* \operatorname{Sq}^4 \iota_n) = A$ *where* d_3 *is the third Bockstein differential as described in section 4.2.3.*

Proof.

1. We show that $d_2(p_1^* \operatorname{Sq}^4 \iota_n) = \gamma$ where $d_2 \colon H^n(X_1; \mathbb{Z}_2) \to H^{n+1}(X_1; \mathbb{Z}_2)$ is the second Bockstein differential defined as in section 4.2.3. Then $\operatorname{Sq}^1 d_2 \operatorname{Sq}^4 \iota_n = 0 = \operatorname{Sq}^1 \gamma$ since the composition of two Bockstein differentials is zero.

   In $H^*(K(\mathbb{Z}, n))$, $\operatorname{Sq}^1 \operatorname{Sq}^4 \iota_n = \operatorname{Sq}^5 \iota_n = \tau(\operatorname{Sq}^2 \operatorname{Sq}^1 \iota_{n+1})$ for $\operatorname{Sq}^2 \operatorname{Sq}^1 \iota_{n+1}$. Therefore by the Bockstein lemma 4.34, $i_1^* d_2 p_1^* \operatorname{Sq}^4 \iota_n = \operatorname{Sq}^1 \operatorname{Sq}^2 \operatorname{Sq}^1 \iota_{n+1} = \operatorname{Sq}^3 \operatorname{Sq}^1 \iota_{n+1}$. We also have $i_1^* \gamma = \operatorname{Sq}^3 \operatorname{Sq}^1 \iota_{n+1}$, and since γ is the only class in $H^{n+5}(X_1)$, we get $d_2(p_1^* \operatorname{Sq}^4 \iota_n) = \gamma$.

2. We have $d_2(p_1^* \operatorname{Sq}^4 \iota_n) = \gamma$ with $\gamma = \tau(\operatorname{Sq}^2 \iota_{n+2})$ for $\operatorname{Sq}^2 \iota_{n+2} \in H^{n+4}(K(\mathbb{Z}_2, n+2))$. Thus we can apply the Bockstein lemma again and yield $i_2^* d_3(p_2^* p_1^* \operatorname{Sq}^4 \iota_n) = \operatorname{Sq}^1 \operatorname{Sq}^2 \iota_{n+2} = \operatorname{Sq}^3 \iota_{n+2} = i_2^*(A)$. Once more since A is the only class in $H^{n+5}(X_2)$, we get $d_3(p_2^* p_1^* \operatorname{Sq}^4 \iota_n) = A$.

$\square$

4.3 Outlook: The Adams spectral sequence

We will conclude with a cursory discussion of the Adams spectral sequence, which supersedes the stable homotopy group calculations we just carried out.

John F. Adams introduces the Adams spectral sequence as a more homological algebra-oriented approach to our calculations in section 4.2.4. The Adams spectral sequence makes use of the fact that for a space X, $H^*(X;\mathbb{Z}_2)$ is a module over the Steenrod algebra. Note that we can also consider $\mathbb{Z}_2$ as a left-module over $\mathcal{A}$ where Sq^0 acts as the identity and all other squares act trivially. Therefore we can consider $\mathrm{Ext}^*_{\mathcal{A}}(H^*(X),\mathbb{Z}_2)$, and we turn this into a bigraded object by considering a grading on Hom with $\mathrm{Hom}^t_{\mathcal{A}}(H^*(X),\mathbb{Z}_2)$ being maps of degree $-t$. Then we make the leap and consider X a spectrum instead. We will not outline how this works, but one can consult [7, Section 5.2]. In any case, for our purposes, we get:

Theorem 4.36 ([10, Theorem 1.2.10])**.** *There is a cohomological spectral sequence converging to the* 2*-component of* $\pi_{n+k}(S^n)$ *for* $k < n-1$ *with*

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{\mathcal{A}}(\mathbb{Z}_2,\mathbb{Z}_2).$$

The groups $E_\infty^{s,t}$ *for* $t-s=k$ *form the associated graded group to a filtration of the* 2*-component of* $\pi_{n+k}(S^n)$*.*

Remark 4.37. The Adams spectral sequence applies more generally for connective spectra with cohomology of finite type. For a suspension spectrum $\Sigma^\infty X$, the E_2 page is $\mathrm{Ext}_{\mathcal{A}}(H^*(X),\mathbb{Z}_2)$. Here we are taking the sphere spectrum $\mathbb{S}=\Sigma^\infty S^0$ (where $\Sigma^n S^0 = S^n$), and then $H^*(S^0)=\mathbb{Z}_2$. We call $H^{s,t}(\mathcal{A}) := \mathrm{Ext}^{s,t}_{\mathcal{A}}(\mathbb{Z}_2,\mathbb{Z}_2)$ the cohomology of the Steenrod algebra $\mathcal{A}$.

Even describing the E_2 page in theorem 4.36 is very difficult, in fact one typically uses another spectral sequence, the May spectral sequence, to do this. There, we see our calculations on the dual Steenrod algebra $\mathcal{A}^*$ come into play.

We will give a partial theoretical sketch of computing the E_2 page, and explain the analogies to our computations in section 4.2.4. Let us just explain this for X a space.

We compute $\mathrm{Ext}^{*,t}_{\mathcal{A}}(H^*(X),\mathbb{Z}_2)$ as follows:

1. Take a free resolution of $H^*(X)$: $\cdots \to F_2 \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0 \xrightarrow{\phi_0} H^*(X) \to 0$
2. Apply $\mathrm{Hom}^t_{\mathcal{A}}(-,\mathbb{Z}_2)$ to this and drop the last term: $\cdots \leftarrow \mathrm{Hom}^t_{\mathcal{A}}(F_2,\mathbb{Z}_2) \xleftarrow{\phi_2^*} \mathrm{Hom}^t_{\mathcal{A}}(F_1,\mathbb{Z}_2) \xleftarrow{\phi_1^*} \mathrm{Hom}^t_{\mathcal{A}}(F_0;\mathbb{Z}_2) \leftarrow 0$
3. Take the homology of this sequence: $\mathrm{Ext}^{r,t}_{\mathcal{A}}(H^*(X),\mathbb{Z}_2) := \ker \phi^*_{r+1}/\operatorname{Im}\phi^*_r$.

A free resolution of $H^*(X)$ means that each F_i is free over $\mathcal{A}$. To find a free resolution of $H^*(X)$, we take what is called an Adams tower:

$$\begin{tikzcd}
X = X_0 & X_1 \arrow[l] \arrow[d, "f_1"] & X_2 \arrow[l] \arrow[d, "f_2"] & X_3 \arrow[l] & \cdots \arrow[l] \\
K_0 \arrow[from=1-1, "f_0"] & K_1 & K_2 & &
\end{tikzcd}$$

One can compare this to our tower (4) at the beginning of section 4.2. There, we mapped into Eilenberg-MacLane spaces to kill off cohomology classes one-by-one. This time we map into wedge products of Eilenberg-MacLane spaces to kill off cohomology all-at-once.

We let K_0 be a wedge of Eilenberg-MacLane spaces $K_0 = \prod_{j>0} K(H^j(X;\mathbb{Z}_2),j)$ such that $f_0\colon X_0 \to K_0$ is surjective on cohomology: $f_0^*\colon H^*(K_0)\to H^*(X_0)$ (f_0 represents the cohomology of X). By using the path space fibration over K_0, we can pull this back along f_0 and obtain X_1, similar to as we did before. Then we repeat, mapping X_1 to $K_1 = \prod_{j>0} K(H^j(X_1;\mathbb{Z}_2),j)$ via f_1 such that f_1^* induces a surjection on cohomology.

Since we can always apply a square of high enough degree to a cohomology class to obtain 0, we shouldn't be able to have $H^*(Y)$ as a free $\mathcal{A}$-module for a space Y. We know from theorem 4.20 that $H^*(K(\mathbb{Z}_2, n))$ is a free $\mathcal{A}$-module up to degree $2n$. We can define $\mathcal{A}$ by taking the limit of $H^*(K(\mathbb{Z}_2, n))$ as n goes to infinity, and we claim that this is "enough" to consider our $H^*(K_i)$ as free $\mathcal{A}$-modules.

Now we show why we want f_i^* surjective. We assume that we are in the stable range in the sense of theorem 4.23, then we have exactness:

$$\cdots \to H^j(K_i) \xrightarrow{f_i^*} H^j(X_j) \to H^j(X_{i+1}) \to H^{j+1}(K_i) \to \cdots$$

moreover the surjectivity of f_i^* ensures that this is

$$\cdots \to H^j(K_i) \twoheadrightarrow H^j(X_j) \xrightarrow{0} H^j(X_{i+1}) \hookrightarrow H^{j+1}(K_i) \to \cdots$$

so we have the short exact sequence

$$0 \to H^j(X_{i+1}) \to H^{j+1}(K_i) \to H^{j+1}(X_i) \to 0$$

and we can replace $H^j(X_{i+1})$ by $H^{j+1}(\Sigma X_{i+1})$ obtaining

$$0 \to H^{j+1}(\Sigma X_{i+1}) \to H^{j+1}(K_i) \to H^{j+1}(X_i) \to 0.$$

We can apply the suspension isomorphism across these short exact sequences and they remain exact. We can then splice things together as follows:

$$\begin{tikzcd}
H^*(\Sigma^3 X_3) \arrow[r] & H^*(\Sigma^2 K_2) \arrow[r] & H^*(\Sigma^2 X_2) \arrow[dll, rounded corners] & \\
H^*(\Sigma^2 X_2) \arrow[r] & H^*(\Sigma K_1) \arrow[r] & H^*(\Sigma X_1) \arrow[dll, rounded corners] & \\
H^*(\Sigma X_1) \arrow[r] & H^*(K_0) \arrow[r] & H^*(X) \arrow[r] & 0
\end{tikzcd}$$

Therefore, we get a free resolution of $H^*(X)$ by letting $F_i = H^*(\Sigma^i K_i)$.

We haven't been properly technical here, but have instead tried to show that the starting points of the Adams spectral sequence can be considered as an extension of the ideas we presented in section 4.2. One can see descriptions of the Adams spectral sequence in [8, Chapter 18], [10, Chapter 3], and [7, Chapter 2].

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