T_2E

Edwina Aylward

Malvern, April 1st 2026

Set up

Let $(K, \mathfrak{m})$ be a local field of odd residue characteristic p.

Let $C_1 \colon y^2 = f_1(x)$, $C_2 \colon y^2 = f_2(x)$ be hyperelliptic curves over K such that

- The leading coefficients c_{f_1} of f_1 and c_{f_2} of f_2 satisfy $\frac{c_{f_1}}{c_{f_2}} \in K^{\times 2}$.
- There is a Galois-equivariant bijection $\phi \colon \{\text{roots of } f_1(x)\} \to \{\text{roots of } f_2(x)\}$ such that $\frac{\phi(r_i)-\phi(r_j)}{r_i - r_j} \equiv 1 \mod \mathfrak{m}$ for all roots $r_i \neq r_j$ of f_1.

In other words, C_1 and C_2 have the same cluster pictures with Galois action, and the lead terms in the π-adic expansions of $r_i - r_j$ and $\phi(r_i) - \phi(r_j)$ agree.

Set up

When this is the case,

M2D2:

- C_1 and C_2 attain semistable reduction over the same extensions of K.
- For $\ell \neq p$, the graded pieces of the filtrations

$$0 \subset (T_\ell \operatorname{Jac} C_i)^t \subset (T_\ell \operatorname{Jac} C_i)^{I_F} \subset T_\ell \operatorname{Jac} C_i$$

  are isomorphic as G_K-modules for $i = 1, 2$, where F is a finite Galois extension over which C_1 and C_2 are semistable.
- $V_\ell \operatorname{Jac} C_1 \simeq V_\ell \operatorname{Jac} C_2$ as G_K-modules for $\ell \neq p$.

Faraggi-Nowell:

- Assuming tame reduction, $\operatorname{Jac} C_1$ and $\operatorname{Jac} C_2$ have equal Néron component groups.

$\star\star$ Note that the ℓ-part of the component group is given by $H^1(I_K, T_\ell \operatorname{Jac} C_i)_{\text{tors}} \star\star$.

Question

Is $T_\ell \operatorname{Jac} C_1 \simeq T_\ell \operatorname{Jac} C_2$ as G_K-modules for $\ell \neq p$?

Zarhin and halving points

We consider $\ell = 2$.

Let $C\colon y^2 = \prod_{i=1}^{2g+1}(x - \alpha_i)$ be a hyperelliptic curve over K. Let $P = (a, b) \in C(\overline{K})$.

In his paper *Division by* 2 *on hyperelliptic curves and Jacobians*, Zarhin shows that the halves of the divisor class $P - (\infty)$ are in 1-to-1 correspondence with choices of square roots:

$$\tau_i = \sqrt{a - \alpha_i} \quad i = 1, \dots 2g+1, \text{ with } \prod \tau_i = -b.$$

Wishful thinking

Since the 2-adic Tate module $T_2 \operatorname{Jac} C$ is built by repeatedly halving points from the 2-torsion, can we use these explicit formulae to directly construct an isomorphism $T_2 \operatorname{Jac} C_1 \xrightarrow{\sim} T_2 \operatorname{Jac} C_2$?

This does work for elliptic curves!

The case of elliptic curves: halving points

Let $E\colon y^2 = (x-\alpha)(x-\beta)(x-\gamma)$ be an elliptic curve over K.

Given a point $P \in E(\overline{K})$, there are exactly four points $Q \in E(\overline{K})$ such that $2Q = P$.

These halves are in 1-to-1 correspondence with triples of square roots $(r_\alpha, r_\beta, r_\gamma)$ such that

$$r_\alpha^2 = x(P) - \alpha, \quad r_\beta^2 = x(P) - \beta, \quad r_\gamma^2 = x(P) - \gamma, \qquad r_\alpha r_\beta r_\gamma = -y(P).$$

If Q corresponds to $(r_\alpha, r_\beta, r_\gamma)$, then

$$x(Q) = x(P) + r_\alpha r_\beta + r_\alpha r_\gamma + r_\beta r_\gamma, \qquad y(Q) = (r_\alpha + r_\beta)(r_\alpha + r_\gamma)(r_\beta + r_\gamma).$$

In particular,

$$x(Q) - \alpha = (r_\alpha + r_\beta)(r_\alpha + r_\gamma),$$

and similarly for permutations of α, β, γ.

The case of elliptic curves: constructing the isomorphism

Let

$$E_1\colon y^2 = (x-\alpha)(x-\beta)(x-\gamma), \quad E_2\colon y^2 = (x-\alpha')(x-\beta')(x-\gamma'),$$

and suppose $\phi(i) = i'$ is a Galois-equivariant bijection preserving the cluster picture and lead terms of root differences.

We construct a G_K-equivariant isomorphism $\varphi\colon T_2E_1 \to T_2E_2$. Inductively construct φ on $E_1[2^n]$ such that

1. $2\varphi(P) = \varphi(2P)$.
2. It is a Galois-equivariant group morphism.
3. For $P \in E_1[2^n]$, $\dfrac{x(\varphi(P)) - r'}{x(P) - r} \equiv 1 \mod \mathfrak{m}$ for $r \in \{\alpha, \beta, \gamma\}$.

For $n = 1$, Let $\varphi((r, 0)) = (r', 0)$ for $r \in \{\alpha, \beta, \gamma\}$, and let $\varphi(\mathcal{O}_{E_1}) = \mathcal{O}_{E_2}$.

Assume that φ has been defined on $E[2^{n-1}]$. Let $P \in E[2^n]$. There is a triple $(r_\alpha, r_\beta, r_\gamma)$ corresponding to P as a half point of $2P$. As

$$\frac{\varphi(x(2P)) - \alpha'}{r_\alpha^2} \equiv 1 \mod \mathfrak{m},$$

we may choose $r'_\alpha = \sqrt{x(\varphi(2P)) - \alpha'}$ such that $\frac{r'_\alpha}{r_\alpha} \equiv 1 \mod \mathfrak{m}$. Similarly define r'_β, r'_γ. Then $\varphi(P)$ is defined as the half point of $\varphi(2P)$ corresponding to $(r'_\alpha, r'_\beta, r'_\gamma)$.

The case of elliptic curves: constructing the isomorphism

- It is clear that $2\varphi(P) = \varphi(2P)$.
- Can show that $\dfrac{r_i' + r_j'}{r_i + r_j} \equiv 1 \mod \mathfrak{m}$ for $i \neq j \in \{\alpha, \beta, \gamma\}$. Thus

$$\frac{x(\varphi(P)) - \alpha'}{x(P) - \alpha} = \frac{r_\alpha + r_\beta}{r_\alpha' + r_\beta'} \cdot \frac{r_\alpha + r_\gamma}{r_\alpha' + r_\gamma'} \equiv 1 \mod \mathfrak{m},$$

  and similarly for β, γ.
- Galois-equivariance: ✓.
- Group morphism: painful, but ✓.

Genus 2 attempts

Let $C\colon y^2 = \prod_{i=1}^5 (x - \alpha_i)$. Let $P = (a, b) \in C(\overline{K})$.

There are 16 divisor classes $\mathfrak{a}$ such that $2\mathfrak{a} = P - (\infty)$. Zarhin shows these correspond to quintuples $\tau = (\tau_1, \dots, \tau_5)$ of square roots:

$$\tau_i^2 = a - \alpha_i, \quad \prod_i \tau_i = -b.$$

In Mumford representation, $\mathfrak{a}_\tau = (c_1, V_\tau(c_1)) + (c_2, V_\tau(c_2)) - 2(\infty)$, where the x-coordinates c_1, c_2 are the roots of:

$$U_\tau(x) = (a - x)^2 + s_2(\tau)(a - x) + s_4(\tau),$$

where s_j is the j-th elementary symmetric polynomial in the τ_i.

As with elliptic curves, can we construct $\varphi\colon T_2 \operatorname{Jac} C_1 \to T_2 \operatorname{Jac} C_2$ by choosing a matching quintuple τ' such that $\tau' \equiv \tau \mod \mathfrak{m}$?
Proceeding naively, the roots of $U_{\tau'}(x)$ and $U_\tau(x)$ should have the same distances from $\{\alpha_i\}$.

Genus 2 attempts

Consider the curve:

$$C_1 \colon y^2 = x(x+11^8)(x+1)(x+4)(x+9)/\mathbb{Q}_{11}.$$

We halve $(0,0)-(\infty)$. Choosing the root quintuple $\tau = (0, 11^4, 1, 2, 3)$ yields $U_\tau(x) = x^2 - s_2(\tau)x + s_4(\tau)$ where $v(s_2(\tau)) = 1$ and $v(s_4(\tau)) = 4$, so the roots of $U_\tau(x)$ have valuations $\{1, 3\}$.

Now, perturb the curve slightly:

$$C_2 \colon y^2 = x(x+11^8)(x+1)(x+4)(x+9+11\cdot 581)/\mathbb{Q}_{11}.$$

We halve $(0,0)-(\infty)$ again using the matching quintuple $\tau' = (0, 11^4, 1, 2, 80)$. Then $U_{\tau'}(x) = x^2 - s_2(\tau')x + s_4(\tau')$ has $v(s_2(\tau')) = 2$ and $v(s_4(\tau')) = 4$. Consequently, the roots of $U_{\tau'}(x)$ now have valuations $\{2, 2\}$.

So trying to match the points appearing in the Mumford representations of $\mathfrak{a}_\tau$ and $\mathfrak{a}_{\tau'}$ appears too naïve.

As of now I'm not sure how to proceed. . .

Why would one want $T_\ell \operatorname{Jac} C_1 \simeq T_\ell \operatorname{Jac} C_2$?

If yes $\implies$ the integral Tate module is a function of the combinatorial data of the roots of the equation defining a hyperelliptic curve.

Consequently, such a result extends to curves C_1/K and C_2/K' over *different local fields* K, K' with the same residue field k, provided one requires:

- Matching cluster pictures with Galois action (including lead terms of root differences).
- An isomorphism $\operatorname{Gal}(K(\operatorname{Jac} C_1[2^\infty])/K) \simeq \operatorname{Gal}(K'(\operatorname{Jac} C_2[2^\infty])/K')$.

This is overkill if we only cared about, for instance, Tamagawa numbers of hyperelliptic curves.

Example

Consider the bihyperelliptic curve $B_{f,g}\colon \{y^2 = f(x), z^2 = g(x)\}/K$, where $f(x)g(x)$ is squarefree. Its Jacobian admits a 2-power degree isogeny to a product of Jacobians of hyperelliptic curves:

$$\phi\colon \operatorname{Jac} B_{f,g} \longrightarrow \operatorname{Jac} C_f \times \operatorname{Jac} C_{fg} \times \operatorname{Jac} C_g.$$

The kernel of this isogeny is explicitly known, thus the 2-part of the Néron component group $\Phi(\operatorname{Jac} B_{f,g})[2^\infty]$ can be computed from $T_2 \operatorname{Jac} C_f$, $T_2 \operatorname{Jac} C_g$, and $T_2 \operatorname{Jac} C_{fg}$.

$\implies$ If the integral Tate modules of hyperelliptic curves depend only on cluster pictures $+\ \epsilon$, then so does $\Phi(\operatorname{Jac} B_{f,g})[2^\infty]$.

Thank you!